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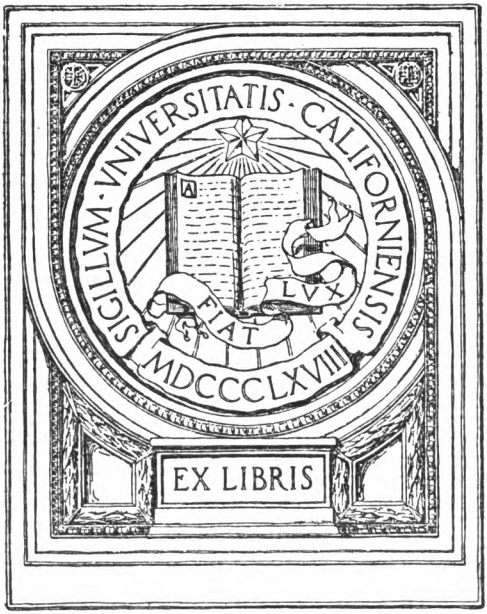

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A TREATISE

OF

PRACTICAL ARITHMETIC,

INTENDED FOR

THE USE OF SCHOOLS.

PART II.

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A TREATISE

OF

PRACTICAL ARITHMETIC,

INTENDED FOR

THE USE OF SCHOOLS;

IN TWO PARTS.

BY ROBERT PATTERSON,

LATE PROFESSOR OF MATHEMATICS AND NATURAL PHILOSOPHY, IN THE UNIVERSITY OF PENNSYLVANIA.

=仕.2=

PITTSBURGH:

PUBLISHED BY R. PATTERSON AND LAMBDIN, ALSO BY M. CAREY AND SON, PHILADELPHIA—COLLINS AND CO., AND COLLINS AND HANNAY, NEW YORK—CUSHING AND JEWETT, AND FIELDING LUCAS, JR., BALTIMORE, AND J. W. PALMER, LEXINGTON.

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1819.

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EASTERN DISTRICT OF PENNSYLVANIA, to wit:

BE IT REMEMBERED, That on the twenty-seventh day of October, in the forty-third year of the independence of the United States of America, A. D. 1818, Joseph Patterson of the said district, hath deposited in this office the title of a book, the right whereof he claims as proprietor in the words following, to wit:

"A Treatise of Practical Arithmetic, intended for the use of Schools; in two parts. By Robert Patterson, late professor of Mathematics and Natural Philosophy, in the University of Pennsylvania."

In conformity to the Act of the Congress of the United States, entitled, "An act for the encouragement of learning, by securing the copies of maps, charts, and books, to the authors and proprietors of such copies, during the times therein mentioned."—And also to the Act entitled, "An Act supplementary to an Act entitled, "An Act for the encouragement of learning by securing the copies of maps, charts, and books, to the authors and proprietors of such copies, during the times therein mentioned," and extending the benefits thereof to the arts of designing, engraving, and etching historical and other prints."

D. CALDWELL,
Clerk of the Eastern District of Pennsylvania.

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☞ IN reviewing the printed copy of this Treatise of Arithmetic, a few *errata* have been discovered, which had escaped detection in examining the proof-sheets, viz.

IN PART FIRST.

Page 54, line 12, read *the gallon wine-measure*, 231 ;
—— 70, last line, for × read +
—— 218, line 10, from bottom, read, *subtract* 2-100*th parts thereof.*

IN PART SECOND.

Page 2, line 3, from bottom, after *Rule*, dele *I.*
—— 5, transpose the first paragraph of this page to the top of page 7 ; and after *Rule*, dele *I.*
—— 13, line 12, for *differences*, read *difference.*
—— 17, line 9, insert *terms* after *greatest.*
—— 18, in theorems 7 and 8, for *n*, read *N.* In theor. 9, dele ÷, and in last line, for ÷ read ×
—— 21, line 5, for *l*, read *L.*
—— 36, line 2 from bottom, read of $4\frac{1}{2}$.
—— 53, line 2 from bottom, for 83, read 11.83.
—— 56, line 5 and 6 from bottom, for 9.64, read 6.94.
—— 78, line 8, read *of which*, 6 *would give the number in question.*

Besides the above, there may still, perhaps, be found some inaccuracies in punctuation, or other slight errata, which can scarce deserve notice : and in the answers to such a multiplicity of questions, subjoined to the respective parts, it is not presumed that some errors may not be discovered.

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EXPLANATION

OF ALGEBRAIC SIGNS OR CHARACTERS, USED OCCASIONALLY IN THE FOLLOWING TREATISE, AS A COMPENDIOUS AND PERSPICUOUS METHOD OF EXPRESSING THE RULES OF OPERATION, OR THEOREMS, IN SOME OF THE HIGHER PARTS OF ARITHMETIC:

To be recurred to, and further explained to the student, in the course of his studies, as occasion may require.

= equals, or equal to, the sign of equality; as $1^{\circ}=60'$
+ more, the sign of addition, as $3+4=7$
— less, the sign of subtraction, as $7-3=4$
⌣ difference, also, a sign of subtraction; as $7\smile 3=4$; or, $3\smile 7=4$
× or . multiplying, the sign of multiplication; as $7\times 4=28$, or $7 \cdot 4=28$
÷ or) (, divided by, or dividing, the sign of division, as $28\div 4=7$, or $4)28(7$
√ square root of; as $\sqrt{36}=6$
∛ cube root of; as $\sqrt[3]{27}=3$
: :: : sign of proportionality; as, 3 : 9 :: 4 : 12
> greater than; as $4>3$
< less than; as $3<4$
∴ ergo, [therefore]

EXPLANATORY NOTES.

1. In expressing rules or theorems in arithmetic, or other branches of mathematics, or in natural philosophy, by algebraic characters, it is usual to represent numbers, whether given or required, by letters of the alphabet, generally taking the *initial* of the denomination of the number represented; as, in interest, *p* for principal, *r* for

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rate per cent. per annum, t for time in years, a for amount, &c.

2. The sign of multiplication is usually omitted between single literal quantities, the letters being set together as letters in a word; thus, ptr signifies $p \times t \times r$.

3. Division is frequently expressed by placing the divisor under the dividend, in form of a vulgar fraction; as, $\frac{12}{3}$ is the same as $12 \div 3$, or $3)12($

4. The *power* of a number or quantity is generally expressed by placing the index of the power on the right, a little above the line; in this case, termed by printers a *superior* figure or letter; as 4^3 signifies the 3d power of 4, r^t signifies the t power of the quantity r, &c.

5. The *root* of a number or quantity is most conveniently expressed by a superior fractional index, as $\frac{1}{2}$ for the square root, $\frac{1}{3}$ for the cube root, &c.

6. A compound quantity, that is, two or more numbers or literal quantities connected by the signs + or —, being enclosed in a parenthesis ( ), signifies that the *whole* compound quantity is affected by the adjoining sign; thus $(7+3-2) \times 5 = 40$. $5.(7+9)^{\frac{1}{2}} = 20$.

7. A *colon* : placed after any term in an algebraic expression is frequently employed to signify that this term *alone* is affected by the following sign, and may be considered as equivalent to the relative pronoun *which*; as $4+5=9 : \times 3 = 27$. $5 \times 6 - 3 = 27 : \div 9 = 3$, &c.

PRAXIS ON THE USE OF THE FOREGOING SIGNS.

$5+3-2=6$

$7 \times 8 \times 2 = 112$

$\frac{16+2}{6} = 3$; or $6)16+2(3$

$(4+6-1) \times 8 = 72$

$(3+4) \times (7-2) = 35$

$(6+4)^2 = 100$

$(100-19)^{\frac{1}{2}} = 9$

Supposing $a=3$, $b=4$, $c=5$, $d=8$

$a+b=7$

$d-c=3$

$ab-d=4$

$\frac{ad}{b} = 6$

$a^2c - \frac{d}{b} = 43$

$(a+b+c)^2 = 144$

$(bc+c)^{\frac{1}{2}} = 5$, &c.

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PRACTICAL ARITHMETIC.

PART II.

OF INVOLUTION.

INVOLUTION is that operation, by which we find any assigned power of a given number. The *power* of a number is the product arising from the continual multiplication of that number any given *number of times.* The number multiplied is called the *root ;* and the number of times that it is a factor, is called the *index* or *exponent* of the power; because it is from this number that the power is denominated. Thus, if the power be the product of *two* equal factors, it is called the *second* power, or *square :* as, $6 \times 6 = 36$, the second power, or square of 6; $7 \times 7 \times 7 = 343$, the third power, or cube of 7; $5 \times 5 \times 5 \times 5 = 625$, the fourth power, or biquadrate of 5, &c. &c.

The following table exhibits the powers of the nine digits, respectively, as far the 9th power.

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Table of the first nine Powers of the nine Digits.

Roots.	Squares.	Cubes.	4th Powers.	5th Powers.	6th Powers.	7th Powers.	8th Powers.	9th Powers.
1	1	1	1	1	1	1	1	1
2	4	8	16	32	64	128	256	512
3	9	27	81	243	729	2187	6561	19683
4	16	64	256	1024	4096	16384	65536	262144
5	25	125	625	3125	15625	78125	390625	1953125
6	36	216	1296	7776	46656	279936	1679616	10077696
7	49	343	2401	16807	117649	823543	5764801	40353607
8	64	512	4096	32768	262144	2097152	16777216	134217728
9	81	729	6561	59049	531441	4782969	43046721	387420489

By the aid of the above table, any high power of a small number, exceeding the limits of the table, may be readily found, as in the following cases.

Case I.

When the root is any of the nine digits, but the power exceeds the limits of the table.

Rule I.

Take two or more powers of the given root, the sum of whose indices shall equal the index of the given

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power, and then the product of these will be the power required.

EXAMPLE.

Find the 15th power of 8.

9th power = 134217728
× 6th power = 262144

= 15th power = 35184372088832

Case II.

When any of the nine digits is a power of the root, as 4 the square of 2, 9 the square of 3, 8 the cube of 2.

Rule.

Find such a power of this digit as shall be equal to the quotient arising from the division of the index of the given power, by the index of the power which the given root is of this digit; and 'tis done.

EXAMPLE.

Find the 27th power of 2; that is, the 9th power of the 3d power.

8 = 3d power of 2;
and the 9th power of 8 = 27 power of 2 = 134217728.
...Ans.

Case III.

When the number to be involved is the product of two single digits, as 42 (= 7 × 6.)

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Rule.

Find, as above, the given power of each factor, and then the product of these will be the power required.

EXAMPLE.

Find the 7th power of 12 [= 2 × 6.]

7th power of 6 = 279936
× 7th power of 2 = 128

= 7th power of 12 = 35831808

EXAMPLES FOR PRACTICE.

1. Raise 4 to the 20th power.
2. Raise 3 to the 30th power.
3. Raise 16 to the 8th power.
4. Raise 24 to the 9th power.

OF EVOLUTION.

Evolution, which is the reverse of involution, is that operation, by which we find, or extract, the root of any given power.

Prob. I. To extract the square root of any given number.

Rule.] 1. Point off the given number into periods of *two* places, beginning at the units' figure.

2. Take the nearest root, not too great, of the left-hand period, which place on the right of the given number, for the first figure of the root.

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Rule I. Take the root of the numerator for a numerator, and that of the denominator for a denominator.

3. Subtract the square of this figure from the first period, and to the remainder annex the next period, for a *resolvend.*

4. Place double the root, on the left of the resolvend, for an *imperfect* divisor.

5. Seek how often this divisor is contained in the resolvend, abating the units' figure, and place the answer for the next figure of the root, and for the units' figure of the *complete* divisor.

6. Multiply this complete divisor by the said figure of the root, subtract the product from the resolvend, and to the remainder annex the next period for a new resolvend.

7. To the last complete divisor, add its units' figure, for a new imperfect divisor, from which find another figure of the root, and thus proceed with the remaining periods of the given number.

NOTE 1. When after subtracting, as above, the product from the last resolvend, nothing remains, the given number is called a *rational* square, but if a remainder be left, it is called a *surd;* and in this case you may continue the operation, annexing periods of cyphers, till the root shall be found to any degree of accuracy required.

NOTE 2. When four or five figures of the root are found by the above process, as many more may be found by common or contracted division, without finding new divisors, generally true to the last figure.

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Ex. 1. Extract the square root of 54990.25.

```
        5'49'90'.25'   (234.5 - - root.
        4
       ----
   43 )149
    3  129
  ----  ----
  464 ) 2090
    4   1856
  ----  -----
 4685 ) 23425
        23425
        -----
          (0)
```

Ex. 2. Extract the square root of 368863, true to four decimal places.

```
          36'88'63'. (607.3409 - - root.
          36
         ----
    1207 )8863
        7 8449
    -----  -----
   12143 ) 41400
           36429
           -----
            4971
            4857
            ----
             114
             109
             ----
```

Prob. II. To extract the square root of any vulgar fraction.

Case I. When both parts of the fraction are rational squares.

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Case II. When both parts of the fraction are not rational squares.

Rule.] Reduce the given fraction to a decimal, and of this decimal find the root to any degree of accuracy required.

Ex. 1. Extract the square root of $\frac{529}{2916}$

529(23	2916(54		
4	25		
43)129	104)416	$\frac{23}{54}$	...Ans.
129	416		
(0)	(0)		

2. Extract the square root of $\frac{7}{9}$

9)7

.7777 &c. (.881917 - - Ans.
64

168)1377
8 1344

1761)3377
1 1761

17629)161677
158661

3066
1763

1303
1224

(79)

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Prob III. To find the square root of the *sum of the squares* of any two given numbers.

Rule.] Divide the square of the less number by double the greater, always increasing the divisor by the two last figures of the quotient, as they occur; and then the quotient added to the greater number will give the square root required*.

Ex. 1. Required the square root of the sum of the squares of 54 and 23.

```
          23
          23
        ----
  108    69
    4   46
  ----  ----
  112)529( 4.69412,+54=58.59412 - - Ans.
    4 448
  ----  ----
  1166)8100
     6 6996
  -----  -----
  11729)110400
      9 105561
  ---------  ---------
  1173,8,4)483900
           469536
           -------
             14364
             11738
             -----
              2626
              2348
              -----
```

* Contracted division may always be used in finding as many of the last figures of the quotient as the places in the divisor, *less two*.

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Ex. 2. Required the square root of the sum of the squares of 1 and 40.

8001)1.0000(40.012498+40=40.012498--Ans.
1 8001

80022 1)99900
2 160044

80,0,2,4 39856
32010

7846
7202

644
640

Prob. IV. To find the square root of the *difference of the squares* of any two given numbers.

Rule.] Divide the square of less number by double the greater, always diminishing the divisor by the two last figures of the quotient, as they occur; and then the quotient subtracted from the greater number will give the square root required.

Ex. 1. Required the square root of the difference of the squares of 21 and 5.

42
6

414)25.00(.60392 from 21=20.39608--Ans.
6 2484

4080 160000
3 122391

40797 37601

407,9,4 887
815

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Ex. 2. Required the square root of the difference of the squares of 82 and 30.

```
164
  56
-----
 159 )900(5.6848 from 82=76.3152--Ans.
----- 795
 1534 ----
    68 )10500
 ------ 9204
         ------
152,7,2 )129600
         122176
         ------
           7424
           6108
           ----
           1316
           1221
           ----
```

Prob. V. To extract the cube root of any given number.

Rule.] 1. Point off the given number into periods of three places, beginning at the units' figure.

2. Take the nearest root, not too great, of the left-hand period, which place on the right of the given number, for the first figure of the root.

3. Subtract the cube of this figure from the first period, and to the remainder annex the next period, for a *resolvend.*

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4. Place triple the square of the root on the left of the resolvend for an *imperfect* divisor.

5. Seek how often this divisor is contained in the resolvend, abating the two right-hand figures; place the answer for the next figure of the root, and annex its square to the right of the imperfect divisor; supplying the place of tens with a cypher, when the square is less than ten.

6. Complete the divisor by adding thereto the product of 30 times the last figure into the preceding figures of the root.

7. Multiply, subtract, and to the remainder bring down the next period, for a new resolvend.

8. The third, and any succeeding figure of the root, is found by seeking how often the last divisor is contained in the resolvend, abating the two right-hand figures.

9. A new divisor is to be found for every new resolvend, thus—to the last complete divisor add the number which completed it, together with twice the square of the last figure of the root, and the sum will be the new defective divisor, with which you are to proceed as before directed*.

* When one more than half the number of figures required in the root (not less than four) is found, according to the rule, then, having found your next imperfect divisor, the remaining figures may be found, generally true to the last, by common, or contracted division.

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Ex. 1. Extract the cube root of 470184984576.

```
                470'184'984'576'(7776 - - root.
                343
               ------
14749)127184
 1470 113533
------ ------
16219  )13651984
------   12564433
1778749 ---------
  16170 )1087551576
-------   1087551576
1794919  ----------
---------      (0)
181118736
   139860
---------
181258596
```

Ex. 2. Extract the cube root of 2, true to 7 places.

```
          2. (1.259921 - -Ans.
          1
 304 ---
  60 ) 1000
----    728
 364  -------
----  )272000
 43225 225125
  1800 --------
------ )46875000
 45025  42491979
------- ---------
 4687581 )4384021
   33750  4279719
-------- --------
 4721331   104302
--------    95105
 4755,2,4,3 -------
              9197
              4751
              -----
```

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Prob. VI. To extract the root of *any given power*, of which the index is a whole number.

GENERAL RULE—BY APPROXIMATION.

1. Beginning at the place of units, point off the given number into periods of as many figures, as far as they will go, as the indexof the power contains units; that is, for the cube root, into periods of three figures, for the biquadrate, or fourth-power root, into periods of four figures, &c.

2. Seek, in the table of powers, the *nearest* root, whether greater or less, to the first or left-hand period; and take the differences between said period, and the proper power of this root, which multiply by the said root, and to the product annex the next period for a resolvend. Call this power, with its root, the *assumed power*, and the *assumed root*, respectively.

3. Multiply the above assumed power by the index more one, and the first period of the given power by the index less one, and call half the sum of these two products your divisor*.

4. Divide the resolvend by this divisor, carrying the quotient to two places of figures.

5. Increase or diminish the first assumed root, according as it was taken too little or too great, by this quotient, and the sum or difference will be a new approximate root, generally true to three places.

6. Proceed with this approximate root in every respect as with the assumed root, carrying the quotient to

* When the index of the power is an odd number, you may multiply by *half* the index more one, and by *half* the index less one, instead of taking half the sum of the products.

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four places of figures; which being applied, by addition or subtraction, to the approximate root before found, will give the root generally true to six or seven places; and, if still greater accuracy should be required, the operation may be repeated*.

Ex. 1. Extract the square root of 2, true to five decimal places.

$2 = 1 \times 2$ (1.
$3 = 3 \times 1$

2)5(2.5)100 (.4

2. 1.4, first app. root.
$5.88 = 3 \times 1.96 = 1.4^2$.
2)7.88

.4
1.4

3.99).56(.01421

1.41421 root required.

* In raising any approximate root to its proper power, contracted multiplication may be used, retaining only one or two places of figures more in the product or power than the places to which the quotient is to be carried; and in finding the quotient you may use contracted division. These contractions will render the operations very easy and expeditious.

When the index of the given power is divisible by *two* or by *three*, it may be depressed to a lower power by extraction of the square or cube root; but the *immediate* extraction of the required root, by the above rule, is generally to be preferred.

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2. Extract the cube root of 25, true to six decimal places.

25 = 1 × 25(3.
54 = 2 × 27

2000
3

79)6000(.076

25 2.924 first app. root.
50=2×24.999545=3d power of app. root (with contraction.)

.000455
2.924

75).0013304(.000018

2.924018 root required.

3. Extract the 5th root of 150, true to five decimal places.

300 = 2 × 150(3.
729 = 3 × 243

93
3

1029)279(.27

2.73 first app. root.

300
454.9194=3× 151.639811 = 5th power of app. root

1.639811
2.73

754.9194)4.476684(.00593

2.72407 root required.

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4. Extract the 6th root of 360, true to five decimal places.

$1800 = 5 \times 360$(3.
$5103 = 7 \times 729$

2)6903 — 369 — 3

3451.5)1107(.32

2.68 1st app. root.

1800
$2593.623 = 7 \times 370.51753 = 268^6$ (contracted.)

2)4393.623 — 10.51753 — 278

2196.811)28.19698(2.68 — 0.01283

2.66717 root req'd.

EXAMPLES FOR PRACTICE, IN THE EXTRACTION OF ROOTS.

1. Extract the square root of 66.5, true to five decimal places.

2. Extract the cube root of 272, true to six decimal places.

3. Find the square root of the *sum* of the squares of 72 and 20.

4. Find the square root of the *difference* of the squares of 72 and 20.

5. Extract the square root of $\frac{49}{100}$.

6. Extract the square root of $\frac{3}{4}$.

7. Extract the 5th root of 316233, true to four decimal places.

8. Extract the 4th root of 1.75, true to six decimal places.

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OF ARITHMETICAL PROGRESSION.

ANY rank of numbers, the terms of which continually increase from the least to the greatest, or decrease from the greatest to the least, by a common difference, is said to be in arithmetical progression; as,

0, 3, 6, 9, 12, 15, 18, &c. increasing by com. diff. 3.
or, 24, 20, 16, 12, 8, 4, 0, &c. decreasing by com. diff. 4.

In a series of numbers in arithmetical progression, the least and greatest are called the two *extremes*, and the *middle* term, when their number is odd, is said to be an *arithmetical mean* between the two extremes, and is equal to half their sum, or that of any two terms equally distant from it. Thus, of the series, 1, 2, 3, 4, 5, 6, 7; 1 and 7 are the two extremes, and 4 the arithmetical mean between them; as it is also, between the terms 3 and 5, 2 and 6.

In the following parts of arithmetic, the rules or theorems will be frequently expressed in algebraic characters, as explained at the beginning of each Part, and which must be now sufficiently intelligible.

In the following theorems,

Let l = the least term.
g = the greatest term.
d = the common difference.
N = the number of terms.
n = the number of terms less one ($= N - 1$.)
s = the sum of the series.

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Now, of the above enumerated particulars, l, g, d, n, s any *three* being given, the other *two* may e found; as is exemplified by the following theorems, which comprehend all the most common and useful cases in this part of arithmetic.

Theorem 1, $g = l + nd$.
2, $g = 2s \div \mathcal{N} - l$.
3, $l = g - nd$.
4, $l = 2s \div \mathcal{N} - g$.
5, $d = (g - l) \div n$.
6, $d = (2s - 2\mathcal{N}l) \div \mathcal{N}n$.
7, $n = (g - l) \div d + 1$.
8, $n = 2s \div (g + l)$.
9, $s = (g + l) \times \div \frac{1}{2}\mathcal{N}$.
10, $s = (g - l + d) \times (g + l) \div 2d$.

EXPLANATORY EXAMPLES.

1. Suppose $l = 2$, $d = 3$, $\mathcal{N} = 9$, cons. $n = 8$, to find g.
Theor. 1, $g = 2 + 3 \times 8 = 26$.
2. Suppose $g = 26$, $d = 3$, $\mathcal{N} = 9$, to find l.
Theor. 3. $l = 26 - 3 \times 8 = 2$.
3. Suppose $g = 26$, $l = 2$, $\mathcal{N} = 9$, to find d.
Theor. 5, $d = (26 - 2) \div 8 = 3$.
4. Suppose $g = 26$, $l = 2$, $d = 3$, to find $\mathcal{N}$.
Theor. 7, $\mathcal{N} = (26 - 2) \div 3 + 1 = 9$.
5. Suppose $g = 26$, $l = 2$, $\mathcal{N} = 9$, to find s.
Theor. 9, $s = (26 + 2) \div \frac{1}{2}9 = 126$.

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EXAMPLES FOR PRACTICE, IN ARITHMETICAL PROGRESSION.

1. Suppose the least term of an arithmetical series = 4, common difference = 3, and number of terms = 27. Required the greatest term, and the sum of all the terms.

2. Supposing a heavy body to fall from a state of rest, near the surface of the earth, it will (without considering the resistance of the air), in the first second of time, fall through a space of 16 feet; in the next second, through a space of 48 feet; in the next, 80 feet, and so on, the spaces described increasing 32 feet every second during the time of its descent. How far would it descend in half a minute? and in what time would it descend through the space of a mile?

3. At what equal distances must 100 stones be placed from each other, so that a person, running at the rate of 5 miles an hour, may in two hours deposit them all, one after another, in a basket 4 yards from the nearest stone?

4. How many strokes will the hammer of a common clock make on the bell, during the space of 12 hours?

5. Required the sum of all the numbers contained in a *Multiplication Table*, extending to 12 times 12.

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OF GEOMETRICAL PROGRESSION.

A RANK of numbers, the terms of which continually increase from the least to the greatest, by a common multiplier, or decrease from the greatest to the least, by a common divisor, is said to be in geometrical progression; as,

1, 2, 4, 8, 16, 32, &c. increasing by the common multiplier 2.

Or, 243, 81, 27, 9, 3, 1, decreasing by the common divisor, 3.

The common multiplier is called the *common ratio*, the least and the greatest terms, as in arithmetical progression, are called the *extremes*, and the middle term, when their number is odd, is said to be a *geometrical mean*, between the two extremes; and is always equal to the square root of their product, or of the product of any two terms equally distant from it. Thus, of the series, 2, 6, 18, 54, 162; 2 and 162 are the two extremes, and 18 the geometrical mean, between them, as it is also between the terms 6 and 54.

In the following theorems—

Let l = the least term,
g = the greatest term,
R = the common ratio,

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r = the common ratio less one ($= R - 1$,)
$\mathcal{N}$ = the number of terms,
n = the number of terms less one ($= \mathcal{N} - 1$,)
s = the sum of the series,
l = logarithm of.*

Now, of the above enumerated particulars, viz. l, g, R, $\mathcal{N}$, s, any three being given, as in arithmetical progression, the other two may be found, as exemplified in the following theorems, which comprehend all the most useful cases in this part of arithmetic.

Theor. 1, $g = l\ R^{n}$. Or, $Lg = Ll + L\ R \times n$.
2, $g = (s\ r + l) \div R$.
3, $l = g \div R^{n}$. Or, $Ll = Lg - L\ R \times n$.
4, $l = Rg - s\ r$
5, $R = (s - l) \div (s - g.)$
6, $LR = (Lg - Ll) \div n$.
7, $\mathcal{N} = (Lg - Ll) \div LR + 1$.
8, $\mathcal{N} = \dfrac{L\,[r \times (s + l)] - Ll}{L\ R}$.
9, $s = (g\ R - l) \div r$.
10, $s = (R^{N} - 1) \div r \times l$.

EXPLANATORY EXAMPLES.

1. Suppose $l = 2$, $R = 3$, $\mathcal{N} = 6$, to find g.
Theor. 1, $g = 2 \times 3^{5} = 486$.
2. Suppose $g = 486$, $R = 3$, $\mathcal{N} = 6$, to find l.
Theor. 3, $l = 486 \div 3^{5} = 2$.

* The nature and arithmetic of logarithms will be fully explained in the following article.

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3. Suppose $s = 728$, $l = 2$. $g = 486$, to find R.
Theor. 5, $R = (728 - 2) \div (728 - 486) = 3$.

4. Suppose $g = 486$, $l = 2$, $R = 3$, to find N.
Theo. 7, $N = (2.686686 - 0.301010) + 0.477121 + 1 = 6$.

5. Suppose $g = 486$, $l = 2$, $R = 3$, to find s.
Theor. 9, $s = (486 \times 3 - 2) \div 2 = 728$.

EXAMPLES FOR PRACTICE, IN GEOMETRICAL PROGRESSION.

1. Suppose the least term of a geometrical series = 3, the common ratio = 2, and the number of terms = 15. Required the greatest term, and the sum of the series.

2. Suppose the least term of a geometrical series = 1, the greatest term = 4096, and the sum of the series = 5461. Required the common ratio, and the number of terms.

3. Suppose a cannon ball to fly in the first second of time 1200 feet, in the next second $\frac{3}{4}$ of that distance, and so on, decreasing in geometrical progression; how far would it fly before it came to a state of rest?

4. Suppose from a hogshead of spirits, containing 100 gallons, 10 gallons were drawn off, and the hogshead filled up with water; that then from this mixture 10 gallons more were drawn off, and so on; it is required to compute how much spirits would remain in the vessel after 100 such operations.

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OF LOGARITHMS.

LOGARITHMS are artificial numbers, so adapted to the natural numbers, that the *sum* of any two logarithms will be the logarithm of the *product* of their corresponding numbers; and consequently the *difference* of two logarithms will be the logarithm of the *quotient* of their corresponding numbers, namely, that of the minuend divided by that of the subtrahend; also, any *multiple*, or any *part* of a logarithm, will be the logarithm of the *power*, or of the *root*, respectively, of its corresponding number, whose index is the multiplier or divisor of the logarithm. Hence, the multiplication, division, involution, and evolution of numbers, may be performed by the addition, subtraction, multiplication, and division of their respective logarithms; as will be fully shown in the following explanatory examples.

If there be two ranks of numbers, one in arithmetical progression beginning with 0, and the other in geometrical progression beginning with 1; then the terms in the former will have the property of logarithms to the corresponding terms in the latter; whatever may be the common difference in the one rank, or the common ratio in the other.

Thus, in the following series, viz.

0, 1, 2, 3, 4, 5, 6, 7, 8, &c.
1, 2, 4, 8, 16, 32, 64, 128, 336, &c,

Or, 0, 2, 4, 6, 8, 10, 12, &c.
1, 3, 9, 27, 81, 243, 729, &c.

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The sum of 2 and 4 = 6, will give the term corresponding to $4 \times 16 = 64$, &c.

Or, $4 + 8 = 12$, will give the term corresponding to $9 \times 81 = 729$, &c.

From the above, it appears, that there may be, as in fact there are, various systems of logarithms, according to the common differences and common ratios of the progressions from which they may be supposed to be formed. If the common difference be 1, and the common ratio 10, then the system of logarithms corresponding will be that in common use, called Briggs's Logarithms.

This system has the following peculiar property, namely, that the integral part, termed the *index* or *characteristic*, of any logarithm, will be always *one less* than the places of whole numbers in its corresponding number; as is evident from the following series.

0, 1, 2, 3, 4, 5, &c. Logarithms.
1, 10, 100, 1000, 10000, 100000, &c. Numbers.

in which the log. of 1 (one place) is 0, the log. of 10 (two places) is 1, that of 100, 2, &c.

The logarithm of 1 being 0, it follows, that the logarithm of a fraction must be less than 0, termed a *negative* number; thus the logarithm of .2 would actually = log. 2 (0.30103) — log. 10 (1.00000) = — 0.69897; and the log. of .02 would = — 1.69897; but it is usual, and more convenient, to write these logarithms $\bar{1}$. 30103, and $\bar{2}$. 30103, and so of other decimal fractions; which are perfectly equivalent to the former, and in which the index *only* is considered as negative, and the

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decimal part as affirmative; the index being always equal to the place of the first significant figure from the decimal point, in the corresponding number.

The calculation of the logarithms of the intermediate numbers between 1 and 10; between 10 and 100; 100 and 1000; &c. constituting a complete table of logarithms, is an operation of considerable difficulty: and as it properly belongs to the higher branches of mathematics, and would be of no real advantage in practical arithmetic, the process is here omitted.

Tables of logarithms are usually calculated to six decimal places, though there are some to seven, or more, and comprehend the logarithms of all numbers not exceeding four, five, or six places. But as the difference of logarithms is very nearly in direct proportion to the difference of their corresponding numbers, when these differences are small; the logarithm of a given number, and, vice versa, the number corresponding to a given logarithm, even when that number exceeds the limits of the table one, two, or three places, may generally be found, with sufficient accuracy, by simple proportion, as in the following examples.

1. Required the logarithm of 643475.

643400 Log. 5.808481
643500 Log. 5.808549

100 : 68 : : 75 : 51

51

5.808532 ....Ans.

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2. Required the number corresponding to the logarithm 3.761584, true to two decimal places.

Log. 3.761552 - - 5775
Log. 3.761627 - - 5776

75 ; 1 : : 32 : .43
5775.

5775.43 ....Ans.

EXPLANATORY EXAMPLES, IN THE ARITHMETIC OF LOGARITHMS.

1. Find the product of 734 multiplying 82.65

724 - - - - - 2.859739
82.65 - - - 1.917243

Ans. 59838 - - - 4.776982

2. Divide 3746.5 by 13.4

3746.5 - - - - - 3.573625
13.4 - - - - - 1.127105

Ans. 279.59 - - - - 2.446520

3. Find a fourth proportional to 37, 41, and 18.4.

37 - - - - - 1.568202, add co-arith.
41 - - - - - 1.612784
18.4 - - - - 1.264818

Ans. 20.389 - - - 1.309400

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4. Raise 17.13 to the third power.

17.13 - - - - - 1.233757
3

Ans. 5026.56 - - - 3.701271

5. Extract the 5th root of 73461.8

5)
73461.8 - - - - 4.866062

Ans. 9.3586 - - - - 0.971212

6. Extract the cube root of .09344

3)
.09344 - - - - $\overline{2}$.970533

Ans. .45378 - - - - $\overline{1}$.656844 *

7. Find a geometrical mean proportional between 17 and 18.

17 - - - - 1.230449
18 - - - - 1.255273

2)2.485722

Ans. 17.493 - - - 1.242861

* In dividing a logarithm having a negative index, not *exactly* divisible by the divisor, you must mentally increase the index by such a number as will make it thus divisible, conceiving the increasing number to be prefixed to the decimal part of the logarithm, as in the above example.

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EXAMPLES FOR PRACTICE, IN THE USE OF LOGARITHMS.

1. Multiply together 3.4, 72.3, and 836.
2. Divide 365 by 21.3, and 21.3 by 365.
3. Raise 1.06 to the 31st power.
4. Extract the 31st root of 6.0881.
5. Extract the square root of .03467.
6. Extract the 5th root of .987654.
7. Find a fourth proportional to 6.2, 12.34, and 19.9.
8. Find a mean proportional between 3, and .0795.

OF COMPOUND INTEREST.

COMPOUND interest, is that which arises from the principal increased by the interest as it becomes due at the end of each year, or other stipulated time of payment*.

* Compound interest, though illegal in most countries, must, nevertheless, be acknowledged to be strictly just; since simple interest, if duly paid according to the conditions of loan, might be employed by the lender so as actually to produce an equivalent to compound interest; and certainly the delinquency of the debtor ought not, in equity, to operate in his own favour, and to the disadvantage of the creditor. Yet, after all, the laws against compound interest may still be justified, on the score of good policy; since they will act as a salutary stimulus on the creditor to prevent too great an accumulation of interest unpaid, which might finally be injurious both to debtor and creditor.

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In the following theorems, which comprehend all the simple cases in compound interest—

Let p = the principal or sum at interest,
t = the time at interest, in years;
R = the ratio, or amount of one pound or dollar, in one year,
a = the amount of principal and interest in t years.
L = logarithm of —

Theor. 1. $a = pR^t$; or, $La = Lp + LR \times t$.

2. $p = \frac{a}{R^t}$; or, $Lp = La - LR \times t$.

3. $R^t = \frac{a}{p}$; or, $LR = \frac{La - Lp}{t}$

4. $R^t = \frac{a}{p}$; or, $t = \frac{La - Lp}{LR}$

The easiest and most expeditious method of solving questions in compound interest is from the logarithmic equations in the above theorems, by means of a table of logarithms: but, when such a table is not at hand, the operation may be greatly facilitated by the two following tables; the use of which will be illustrated in the following explanations, and examples.

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Table A, *showing the amount of one pound or dollar, from ¼ year to 40 years.*

Years.	4 per cent.	5 per cent.	6 per cent.
¼	1.009853	1.012272	1.014674
½	1.019804	1.024695	1.029563
¾	1.029852	1.037270	1 044671
1	1.0400000	1.0500000	1.0600000
2	1.0816000	1.1025000	1 1236000
3	1.1248640	1.1576250	1.1910160
4	1.1698585	1.2155062	1.2624769
5	1.2166529	1.2762815	1 3382256
6	1.2653190	1.3400956	1.4185191
7	1.3159317	1.4071004	1 5036302
8	1.3685690	1.4774554	1.5938480
9	1 4233118	1.5513282	1.6894789
10	1.4802442	1.6288946	1.7908476
11	1.5394540	1 7103393	1.8982985
12	1.6010322	1.7958563	2.0121964
13	1.6650735	1.8856491	2.1329282
14	1 7316764	1.9799316	2.2609039
15	1.8009435	2.0789281	2.3965581
16	1.8729812	2.1828745	2.5403517
17	1.9479005	2.2920183	2 6927727
18	2.0258161	2.4066192	2.8543391
19	2.1068491	2.5269502	3 0255995
20	2.1911231	2.6532977	3.2071355
21	2.2787680	2.7859625	3.3995636
22	2.3699187	2.9252607	3.6035374
23	2.4647155	3.0715237	3 8197496
24	2.5633041	3.2250999	4.0489346
25	2.6658363	3.3863549	4.2918707
26	2.7724697	3 5556726	4.5493829
27	2.8833685	3.7334563	4 8223459
28	2.9987033	3.9201291	5.1116867
29	3.1186514	4.1161356	5.4183879
30	3.2433975	4.3219423	5.7434912
31	3.3731334	4,5380394	6.0881007
32	3.5080587	4.7649414	6.4533867
33	3 6483811	5.0031885	6.8405899
34	3.7943163	5.2533479	7.2510253
35	3.9460889	5 5160152	7.6860868
36	4.1039325	5.7918101	8 1472520
37	4.2680898	6.0814069	8 6360671
38	4.4388134	6.3854772	9.1542523
39	4.6163659	6 7047511	9.7035074
40	4.8010206	7.0399887	10 2857178

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EXPLANATION OF THE FOREGOING TABLE.

In this table, the numbers in the several columns express the respective ratios raised to a power whose index equals the corresponding number of years or quarters in the adjoining left-hand column. Thus, in the column 6 *per cent.*, the number 3.2071355, is the 20th power of 1.06; or R^{20}, &c. expressed by R^t in the foregoing theorems.

NOTE 1. When the time exceeds the limits of the table; take two or more times from the table, whose sum equals the time required, and then the product of their corresponding tabular numbers will be the number, or power of the ratio required. Thus, supposing the time to be 50 years, at 6 per cent., I multiply together the two numbers, 10.2857178 and 1.7908476, corresponding, respectively, to 40 years and 10 years, under the given ratio, and the product, 18.4201541, will be the number required, or 1.06^{50}*.

NOTE 2. When the time is in years and quarters; then, multiply together the tabular numbers corresponding to the years and quarters, respectively, and the product will be the number or power of the ratio required. Thus, supposing the time to be 4 years and 3 quarters, at 6 per cent., I multiply together 1.2624769 and 1.044671, and the product, 1.318872, will be the number required, or $1.06^{4\frac{3}{4}}$.

* In this, and in all other similar cases, *contracted* multiplication, should be used, retaining only *two*, or at most *three*, decimal places more than the places of whole numbers in *p*, the principal.

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NOTE 3. When the payments of interest are conditioned to be half-yearly, or quarter-yearly; that is, paying half or quarter of the annual interest; then, the half or fourth of the fractional part of the ratio, together with the unit, is to be raised to a power equal to the half-years or quarter-years contained in the given time. Thus; supposing the payments quarter-yearly, at 6 per cent. and the time 20 years; the corresponding number will be the 80th power of 1.015 = 3.29066; which is to be considered as R^t.

NOTE 4. When the time, besides years, or years and quarters, contains some part of a quarter, or of a month; then, the tabular number for the years, or years and quarters, is to be increased by the proper proportional multiple or part of the *simple* interest for one month, at the given rate. Thus, supposing the given time 4 years, 1 month, and 3 days, at 6 per cent. the tabular number will be found as follows,—

4 years - - - - - -	1.262477
1 month - - - - - -	.005000 ($\frac{1}{12}$ of .06.)
3 days - - - - - - -	.000500 ($\frac{1}{10}$ of .005.)
Tab. num.	1.267977 = R^t.

EXPLANATORY EXAMPLES.

1. Suppose p = 10, R = 1.06, t = 12; to find a.

Theor. 1, La = 1.000000 (L10) + 0.025306 × 12 (L 1.06 × 12) = 1.303672 (L 20.122, = a)

Or, per Tab. A, 2.0121964 × 10 = 20.122, = a.

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2. Suppose $a = 20.122$, $R = 1.06$, $t = 12$; to find p.
Theor. 2, $Lp = 1.30672$ (L 20.122) $- 0.025306 \times 12$ (L 1.06×12) $= 1.000000$ (L 10, $= p$.)
Or, per Tab. A, $20.122 \div 2.121964 = 10$, $= p$.

3. Suppose $a = 20.122$, $p = 10$, $t = 12$, to find R.
Theor. 3, $LR = [1.303672$ (L 20.122) $- 1.000000$ (L 10)$] \div 12 = 0.025306$ (L 1.06, $= R$).
Or, per Tab. A, 20.122 ÷ 10 = 2.0122, opposite 12 and under 6 per cent. or, $1.06 = R$.

4. Suppose $a = 20.112$, $p = 10$, $R = 1.06$; to find t.
Theor. 4, $t = [1.303672$ (L 20.122) $- 1.000000$ (L 10)$] \div 0.02536$ (L 1.06) $= 12$.
Or, per Tab. A, 20.122 ÷ 10 = 2.0122, under 6 per cent. and opposite 12 = t.

EXAMPLES FOR PRACTICE, IN COMPOUND INTEREST.

1. Required the amount of $500 in 4 years, at 6 per cent.

2. Required the amount of $1000 in 5½ years, at 5 per cent.

3. Required the amount of $750 in 7 years and 36 days, at 5 per cent.

4. What principal, at 4 per cent. would, in 10 years amount to $800?

5. What principal, at 4 per cent. payable quarter-yearly, would, in 6 years, amount to $2000?

6. At what rate per cent. per annum would $450 amount to $520.93 in 3 years?

7. In what time would $600 amount to $757.49, at 6 per cent. per annum?

8. In what time would a sum of money double itself, at 6 per cent. per annum?

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OF ANNUITIES IN ARREAR AT SIMPLE INTEREST*.

In the following theorems,

Let a = the annuity, or yearly payment.

r = the ratio, or interest of £1, or $1 for one year, at any given ratio, viz. .03, for 3 per cent. .04, for 4 per cent.; .05, for 5 per cent.; .06, for 6 per cent.; &c.

T = the time, in years, for which the annuity is in arrear.

$t = T - 1$.

$m = \frac{1}{r} - \frac{1}{2}$.

Theor. 1, $a = \frac{1}{2} Tutr + Tu$.

2, $u = a \div (\frac{1}{2} Ttr + T.)$

3, $r = (a - Tu) \div \frac{1}{2} Tut$.

4, $T = (2a \div ur + m^2)^{\frac{1}{2}} - m$.

EXPLANATORY EXAMPLES.

1. Suppose $u = 50, r = .05, T = 12$; to find a.
   Theor. 1, $a = \frac{1}{2} 12 \times 50 \times 11 \times .05 + 12 \times 50 = 765$.
2. Suppose $a = 765, r = .05, T = 12$; to find u.
   Theor. 2, $u = 765 \div (\frac{1}{2} 12 \times .05 \times 11 + 12) = 50$.

* Annuities are usually computed at compound interest; yet, as this, is not always the case, annuities at simple interest, are here inserted.

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3. Suppose $a = 765$, $u = 50$, $T = 12$; to find r.

Theor. 3, $r = (765 - 12\times50)\div(\frac{1}{2}\ 12\times50\times11) = .05$.

4. Suppose $a = 765$, $u = 50$, $r = .05$, consequently, $m = 19.5$; to find T.

Theor. 4, $T = (2\times765 \div .05\times50 + 380.25)^{\frac{1}{2}} - 19.5 = 12$.

EXAMPLES FOR PRACTICE, IN ANNUITIES IN ARREAR, AT SIMPLE INTEREST.

1. If an annuity of $140 be forborne 5 years, what will be the amount at 5 per cent.?

2. What annuity, at 6 per cent. will amount to $1000 in 7 years?

3. At what rate per cent. will $560 amount to $4625.60 in 7 years?

4. In what time will an annuity of $70, amount to $389.375, at 5 per cent.?

OF THE PURCHASE, OR PRESENT-WORTH OF ANNUITIES, AT SIMPLE INTEREST.

In the following theorems,

Let p = the present worth,

u, r, T, and t, the same as in the last article,

$$m = \frac{1}{r} - \frac{p}{u} - \frac{1}{2}.$$

Theor. 1, $p = (Trt + 2t) \div (2\ Tr + 2) \times u$.

2, $u = (2\ Tpr \times 2\ p) \div (Trt + 2\ t)$.

3, $r = (2\ Tu - 2\ p) \div (2\ Tp - Tut)$.

4, $T = (2\ p \div ru + m^2)^{\frac{1}{2}} - m$.

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EXPLANATORY EXAMPLES.

1. Suppose $u = 50, r = .06, T = 8$, consequently, $t = 7$; to find p.

   Theor. 1, $p = (8 \times .06 \times 7 + 16) \div (16 \times .06 + 2) \times 50 = 327{,}03$, nearly.

2. Suppose $p = 327.03$, $r = .06$, $T = 8$; to find u.

   Theor. 2, $u = (2 \times 327.03 \times 8 \times .06 + 2 \times 327.03) \div (8 \times .06 \times 7 + 16) = 50$.

3. Suppose $p = 327.03$, $u = 50$, $T = 8$; to find r.

   Theor. 3, $r = (16 \times 50 - 2 \times 327.03) \div (2 \times 327.03 \times 8 - 8 \times 50 \times 7) = .06$.

4. Suppose $p = 327.03$, $u = 50$, $r = .06$, consequently, $m = 9.62$; to find T.

   Theor. 4, $T = (2 \times 327.03 \div 50 \times .06 + 9.62^2)^{\frac{1}{2}} - 9.62 = 8$.

EXAMPLES FOR PRACTICE, IN THE PURCHASE OF ANNUITIES, AT SIMPLE INTEREST.

1. What is the present worth of $50 per annum, to continue 5 years, at 6 per cent.?

2. What annuity, to continue 7 years, at 4 per cent., would be worth $245?

3. Suppose I purchase a rent of $160 per annum, to continue 5 years, for $689.23; what rate per cent. per annum, will I have on my purchase money?

4. For what time may a pension of $300 per annum, be bought for $1334.69, to receive at the rate $4\frac{1}{2}$ ($r = .045$) per cent. per annum, on the purchase-money?

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OF ANNUITIES IN REVERSION, AT SIMPLE INTEREST.

In the following theorems,

Let u = the annuity, r = the ratio, } as in the last article.
T = the time of continuance of the reversionary annuity.
t = the time before the reversionary annuity commences.
P = the present worth of u in time T.
p = present worth of P in time t, = present worth of reversionary annuity.
a = amount of p in time t.

Theor. 1. $\begin{cases} P = (TTr - Tr + 2T) \div (2Tr + 2) \times u. \\ p = P \div (tr + 1). \end{cases}$

Theor. 2. $\begin{cases} a = (tr + 1) \times p. \\ u = (Tr + 1) \div (TTr - Tr + 2T) \times 2a. \end{cases}$

EXPLANATORY EXAMPLES.

1. Suppose $u = 30, r = .04, T = 3, t = 2$; to find p.

By theor. 1. $\begin{cases} P = (.04 \times 9 - .12 + 6) \div (.24 + 2) \\ \qquad \times 30 = 83.57. \\ p = 83.57 \div 1.08 = 77.38. \end{cases}$

2. Suppose $p = 77.38, t = 2, T = 3, r = .04$; to find u.

By theor. 2. $\begin{cases} a = 1.08 \times 77.38 = 83.57. \\ u = 1.12 \div (.36 - .12 + 6) \times 2 \times 83.57 \\ \qquad = 30. \end{cases}$

EXAMPLES FOR PRACTICE, IN REVERSIONARY ANNUITIES, AT SIMPLE INTEREST.

1. A minor of 16 years of age has a legacy left to him or his heirs, of an annuity, of $100, to continue for 8

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years after he becomes of age—21 years. What is the legacy worth in ready money; allowing the purchaser at the rate of 4 per cent. per annum, on his purchase-money?

2. What annuity to continue 7 years after the expiration of 4 years, may I purchase for $1689\frac{1}{2}$, allowing interest at the rate of 5 per cent. per annum?

OF ANNUITIES IN ARREAR, AT COMPOUND INTEREST.

In the following theorems,

Let u = the annuity.

R = the ratio or amount of £1 or $1, in one year, as in compound interest.

r = the interest of £1 or $1, in one year $= R - 1$.

t = the time for which the annuity is in arrear.

R^t = the ratio involved to the t power, } Found in Table A, foregoing.

$m = R^t - 1$. - - }

a = the amount of annuity, at the end of t years.

p = the present worth of the annuity, to continue for any time t, at any rate R.

L = logarithm of.

Theor. 1. $a = mu \div r$; or, $La = Lm + Lu - Lr$.

Theor. 2. $u = ar \div m$; or, $Lu = La + Lr - Lm$.

Theor. 3. $t = [L(ar + u) - Lu] \div LR$.

Theor. 4. $p = mu \div R^t r$; or, $Lp = Lm + Lu - L\,R \times t - Lr$.

Theor. 5. $u = R^t rp \div m$; or, $Lu = LR \times t + Lr + Lp - Lm$.

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EXPLANATORY EXAMPLES.

1. Suppose $u = 10$, $R = 1.05$, $t = 4$, consequently, $r = .05$, $R^t = 1.215506$ (Tab. A) and $m = .215506$; to find a.

By theor. 1
$$\left.\begin{array}{ll} Lm = & \overline{1}.333447 \\ Lu = & 1.000000 \end{array}\right\} + \qquad \begin{array}{ll} Lr = & \overline{2}.698970 \\ La = & 1.634477 \end{array} - \quad \text{and } a = 43.1.$$

2. Suppose $a = 43.1$. $t = 4$, $R = 1.05$, cons. $r = .05$, and m (from Tab. A) $= .215506$; to find u.

By theor. 2.
$$\left.\begin{array}{ll} La = & 1.634477 \\ Lr = & \overline{2}.698970 \end{array}\right\} + \qquad \begin{array}{ll} Lm = & \overline{1}.333447 \\ Lu = & 1.000000 \end{array} - \quad \text{and } u = 10.$$

3. Suppose $a = 43.1$, $R = 1.05$, $u = 10$; to find t.

By theor. 3.
$$\begin{array}{ll} L(ar + u) = & 1.084755 \\ Lu = & 1.000000 \\ LR = .021189) & .084755 (4 = t. \end{array}$$

4. Suppose $u = 10$, $R = 1.05$, $t = 4$, cons. $m = .215506$; to find p.

By theor. 4.
$$\left.\begin{array}{ll} Lm = & \overline{1}.333447 \\ Lu = & 1.000000 \end{array}\right\} + \qquad \left.\begin{array}{ll} LR \times 4 = & 0.084755 \\ Lr = & \overline{2}.698970 \end{array}\right\} - \qquad \begin{array}{ll} Lp = & 1.649722 \end{array} ; \text{ and } p = 35.459.$$

5. Suppose $p = 35.459$, $R = 1.05$, $t = 4$, cons. $m = .215506$; to find u.

By theor. 5.
$$\left.\begin{array}{ll} LR \times 4 = & 0.084755 \\ Lr = & \overline{2}.698970 \\ Lp = & 1.649722 \end{array}\right\} + \qquad \begin{array}{ll} Lm = & \overline{1}.333447 \\ Lu = & 1.000000 \end{array} - \quad \text{and } u = 10.$$

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All the above examples, except the 3d, may also be wrought without logarithms, by the first theorems, respectively.

EXAMPLES FOR PRACTICE, IN ANNUITIES IN ARREAR, AT COMPOUND INTEREST.

1. What sum will $300 amount to in 4 years, at 5 per cent.?

2. What annuity 5 years in arrear, at 4 per cent., will amount to $2708.16?

3. In what time would a salary of $400 per annum, amount to $3357.54 at 6 per cent.?

4. What must be the discount (abatement) on a lease for 4 years at $500 per year, when payment is made in advance (present worth), supposing compound interest at the rate of 3 per cent. per annum?

5. What annuity, to continue 6 years, may be purchased for $1015.15; so that the purchaser may have 5 per cent. per annum, compound interest, on his purchase-money?

OF REVERSIONARY ANNUITIES, AT COMPOUND INTEREST.

In the following theorems,

Let a = the annuity.

R = the ratio, or amount of £1 or $1, for one year.

r = the interest of £1 or $1, for one year $= R - 1$.

T = the time of reversion, or time before the annuity commences.

t = the time of continuance of the annuity after it commences.

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$R^t - 1 = m$, found in Tab. A.

w = the whole time of reversion and continuance, $= T + t$.

p = the present worth of the reversionary annuity.

L = logarithm of.

Theor. 1. $p = mu \div R^w r$; or, $Lp = Lm + Lu - LR \times w - Lr$.

2. $u = pR^w r \div m$; or, $Lu = Lp + LR \times w + Lr - Lm$.

3. $w = (Lu + Lm - Lp - Lr) \div LR$.

EXPLANATORY EXAMPLES.

1. Suppose $u = 10$, $t = 4$, T 6, $R = 1.05$, cons. $w = 10$, and $m = .215506$; to find p.

By theor. 1.
$$\left.\begin{array}{ll} Lm = & \bar{1}.333446 \\ Lu = & 1.000000 \\ LR \times w = & 0.211893 \\ Lr = & \bar{2}.698970 \\ Lp = & \overline{1.422583} \end{array}\right\} \begin{array}{l} + \\ + \\ + \\ - \end{array}$$
; and $p = 70.205$.

2. Suppose $p = 70.205$, $R = 1.05$, $t = 4$, $T = 6$, cons. $w = 10$, and $m = .215506$; to find u.

By theor. 2.
$$\left.\begin{array}{ll} Lp = & 1.422583 \\ LR \times w = & 0.211893 \\ Lr = & \bar{2}.798970 \\ Lm = & \bar{1}.333446 \\ Lu = & \overline{1.000000} \end{array}\right\} \begin{array}{l} + \\ + \\ + \\ - \end{array}$$
; and $u = 10$.

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3. Suppose $u = 10$, $p = 70.205$, $R = 1.05$, $t = 4$, cons. $m = .215506$; to find T.

By theor. 3.
$$\left.\begin{array}{ll} Lu = & 1.000000 \\ Lm = & \overline{1}.333446 \end{array}\right\} + \qquad \left.\begin{array}{ll} Lp = & 1.422583 \\ Lr = & \overline{2}.698970 \end{array}\right\} -$$
$$LR = 0.021189)0.211893(10 = w; \text{ cons. } T = 6.$$

All the above examples, except the last, may also be wrought without logarithms, by the first theorems, respectively.

EXAMPLES FOR PRACTICE, IN REVERSIONARY ANNUITIES, AT COMPOUND INTEREST.

1. What is the present worth of an annuity of $320 to continue 7 years, but not to commence till the expiration of 4 years; reckoning compound interest on the purchase-money, at the rate of 5 per cent. per annum?

2. What annuity or yearly rent, to commence 2 years hence, and then to continue 4 years, may be purchased for $643.26; interest being reckoned at 5 per cent.?

3. The yearly rent of $200, to commence 2 years hence, is sold, at 4 per cent., for $1244.96. How long is this annuity to continue?

The calculations in all the forgoing cases of annuities at compound interest, may be greatly facilitated by the aid of the following Tables, of which there follows an explanation, with explanatory examples.

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TABLE B. Showing the amount of 1l. or dol. annuity.

year	4 per cent.	5 per cent.	6 per cent.	year.
1	1.	1.	1.	1
2	2.04	2.05	2.06	2
3	3.1216	3.1525	3.1836	3
4	4.246464	4.310125	4.374602	4
5	5.416322	5.525631	5.637093	5
6	6.632975	6.801913	6.975318	6
7	7.898294	8.142008	8.393837	7
8	9.214226	9.549109	9.897468	8
9	10.782795	11.026564	11.491316	9
10	12.006107	12.577892	13.180795	10
11	13.486351	14.206787	14.971643	11
12	15.025805	15.917126	16 869942	12
13	16.626838	17.712983	18.882138	13
14	18.291911	19.598632	21.015066	14
15	20.023588	21.578563	23.275971	15
16	21.824531	23.657492	25.672528	16
17	23.697512	25.840366	28.212881	17
18	25.645413	28.132385	30.905653	18
19	27.671229	30.539004	33.759993	19
20	29.778078	33.065954	36.785592	20
21	31.969202	35.719252	39.992728	21
22	34.24797	38.505214	43.392291	22
23	36.617888	41.430475	46.995828	23
24	39.082604	44.501999	50.815578	24
25	41.645908	47.727099	54.864513	25
26	44.311745	51.113454	59.156383	26
27	47.084214	54.669126	63 705766	27
28	49.967582	58.402583	68.528112	28
29	52.966286	62.322712	73.639798	29
30	56 084938	66.438847	79.058186	30
31	59.328335	70.76079	84.801677	31
32	62.701469	75.298829	90.889778	32
33	66.209527	80.063771	97.343165	33
34	69.857904	85.066959	104.183754	34
35	73.652225	90.320307	111.43478	35
36	77.598314	95.836323	119.120867	36
37	81.702246	101.628130	127.268118	37
38	85.970336	107.709546	135.904206	38
39	90.40915	114.095023	145.058458	39
40	95.025516	120.799774	154.721966	40

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TABLE C. Showing the present worth of 1*l. or dol. annuity, for any number of years, from* 1 *to* 40.

year	4 per cent.	5 per cent.	6 per cent.	year.
1	0.96154	0.95231	0.94339	1
2	1.88609	1.85941	1.83339	2
3	2.77509	2.72325	2.67301	3
4	3.62989	3.54595	3.4651	4
5	4.45182	4.32948	4.21236	5
6	5.24214	5.07569	4.91732	6
7	6 40205	5.78637	5.58238	7
8	6.73274	6.46321	6.20979	8
9	7.43533	7.10782	6.80169	9
10	8.11089	7.72173	7 36008	10
11	8.76048	8.30641	7.88687	11
12	9.38507	8.86325	8.38384	12
13	9.98565	9.39357	8.85268	13
14	10.56312	9.89864	9.29498	14
15	11 11839	10.37965	9 71225	15
16	11.65229	10.83777	10.10589	16
17	12.16567	11.27407	10.47726	17
18	12.65929	11.68958	10.8276	18
19	13.13394	12.08532	11.15811	19
20	13.59032	12.46221	11.46992	20
21	14.02916	12.82115	11 76407	21
22	14.45111	13.163	12.04158	22
23	14.85684	13.48857	12.30338	23
24	15.24696	13.79864	12.55035	24
25	15.62208	14.09394	12.78335	25
26	15.98277	14.37518	13.00316	26
27	16.32959	14.64303	13.21053	27
28	16.66306	14.89813	13.40616	28
29	16.98371	15.14107	13.59072	29
30	17.29203	15.37245	13.76483	30
31	17.58849	15.59281	13.92908	31
32	17.87355	15.80268	14.08404	32
33	18.14764	16.00255	14.23023	33
34	18.4112	16.1929	14.36814	34
35	18.66461	16.37419	14.49825	35
36	18.90828	16.54685	14.62098	36
37	19.14258	16.71129	14.73678	37
38	19.36786	16.86789	14.84602	38
39	19.58448	17.01704	14.94907	39
40	19.79277	17.15909	14.92640	40

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Multipliers for half-yearly and quarter-yearly payments, applicable to the above tables.

TABLE D.

Rate per cent.	*Half-yearly payments.*	*Quarterly payments.*
4	1.009902	1.014877
5	1.012348	1.018539
6	1.014781	1.022257

EXPLANATION.

1. The numbers in the respective columns of Table B, exhibit the element $(R^t - 1) \div r$, in the above theorems; thus 3.1836 found opposite 3 years and under 6 per cent. = .191016 (R^t—1 taken from Tab. A.) ÷ .06.

2. The numbers in Table C, are respectively those in Table B divided by R^t contained in Table A: thus 2.67301, found opposite 3 years, and under 6 per cent. 3.1836 ÷ 1.191016 (R^t in Table B.)

3. The numbers in Tab. D are factors, by which if you multiply the numbers in the above tables, then, the result of the operation with such increased tabular numbers, will give the *amount* or *present worth* of the annuity, when the the payments are half-yearly, or quarterly, respectively: thus 3.1836 × 1.014781 will give 3.230656, the tabular number for the present worth of *L*.1, or $1 annuity to continue 3 years at 6 per cent., the payments being half yearly.

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Making use of the above tables, and putting

$a' =$ any tabular number in table B, and

$p' =$ any tabular number in table C, and all the other symbols as before, the several theorems will be expressed as follows.

I. FOR ANNUITIES IN ARREAR AT COMPOUND INTEREST.

Theor. 1. $a = a'u$

2. $u = a \div a'$

3. $a' = a \div u$; then, either R or t being known, the other may be found, by inspection, in Table B*.

II. FOR THE PURCHASE OR PRESENT WORTH OF ANNUITIES, AT COMPOUND INTEREST.

Theor. 1. $p = p'u$

2. $u = p \div p'$

3. $p' = p \div u$; then, either R or t being known, the other may be found, by inspection, in Table C*.

* When the tabular number a', p', or d, resulting from the operation as above, cannot be exactly (or very nearly) found in its proper table (B or C), which will generally be the case, you may apply the simple proportional part of the difference, which will give t or R sufficiently near the truth. For example, suppose a', in Theor. 3. for annuities in arrear, should come out 31.755, the rate being 5 per cent., and t required; then, in table B, I find under 5 per cent., the two numbers next greater, and next less to be 30.539 and 33.066 (using but three places of decimals) opposite to 19 and 20, in the column of years; therefore say, as 33.066—30.539 : 12 months : : 31.755—30.539 : 5.8, hence t will = 19 yrs. 5.8 months, sufficiently near.

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III. FOR REVERSIONARY ANNUITIES, AT COMPOUND INTEREST.

Let d = the difference of two numbers in table C, under the given rate per cent., one opposite t, the time of reversion, or that before the annuity commences; and *the other* opposite the whole time of reversion and continuance $(T + t)$, and let the other symbols remain as before.

Theor. 1. $p = du$.

2. $u = p \div d$.

3. $d = p \div u$; then, if R be given, and either T or t, the other may be found, or if T and t be both given, R may be found by inspection in Table C*.

☞ The examples for practice under the former theorems, may serve for a like purpose under the above.

OF PERPETUITIES.

A PERPETUITY is an annual, or other periodical payment, to continue during an unlimitted time.

Case I. When the perpetuity commences from the present time.

In the following theorems,

Let u = the perpetuity.

r = the interest of $L.1$ or $\$1$ for a year.

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p = the present worth of the perpetuity, allowing the purchaser the stipulated interest on his purchase-money.

Theor. 1. $p = u \div r$.

2. $u = pr$.

3. $r = u \div p$.

EXPLANATORY EXAMPLES.

1. Suppose $u = 10$, $r = .05$; to find p.

By theor. 1. $p = 10 \div .05 = 200$.

2. Suppose $p = 200$, $r = .05$; to find u.

By theor. 2. $u = 200 \times .05 = 10$.

3. Suppose $u = 10$, $p = 200$; to find r.

By theor. 3. $r = u \div 200 = .05$.

The above theorems are so plain and easy, that examples for practice are deemed unnecessary.

Case II When the perpetuity is in reversion, or to commence at a future time, at compound interest.

In the following theorems,

Let u = the perpetuity.

t = the time of reversion, before the perpetuity commences.

R = the amount of $L.1$ or $£1$ for one year, at any given rate.

$r = R - 1$.

p = the present worth of the reversionary perpetuity.

Theor. 1. $p = u \div R^t r$; Or, $Lp = Lu - LR \times t - Lr$.

2. $u = p R^t r$; Or, $Lu = Lp + LR \times t + Lr$.

3. $t = (Lu - Lp - Lr) \div LR$.

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EXPLANATORY EXAMPLES.

1. Suppose $u = 10$, $t = 3$, $R = 1.05$, cons. R^t (per Tab. A) $= 1.157625$; to find p.

By theor. 1.
$$\left.\begin{array}{ll} Lu = & 1.000000 + \\ LR \times 3 = & 0.063567 \\ Lr = & \bar{2}.698970 \end{array}\right\} -$$
$$Lp = \quad 2.237463; \text{ and } p = 172.8.$$

2. Suppose $p = 172.8$, $t = 3$, $R = 1.05$, cons. R^t (per Tab. A) $= 1.157625$; to find u.

By theor. 2.
$$\left.\begin{array}{ll} Lp = & 2.237463. \\ LR \times 3 = & 0.063567 \\ Lr = & \bar{2}.698970 \end{array}\right\} +$$
$$Lu = \quad 1.000000; \text{ and } u = 10.$$

3. Suppose $u = 10$, $p = 172.8$, $R = 1.05$; to find t.

By theor. 3.
$$\left.\begin{array}{ll} Lu = & 1.000000 + \\ Lp = & 2.237453 \\ Lr = & \bar{2}.698970 \end{array}\right\} -$$
$$LR = .021189) \ .063567(3 = t.$$

EXAMPLES FOR PRACTICE, IN REVERSIONARY PERPETUITIES, AT COMPOUND INTEREST.

1. Required the present value of a freehold estate of £600 per annum, to commence 2 years hence; allowing 6 per cent. per annum compound interest on the purchase-money.

2. What yearly income, in perpetuity, to commence at the expiration of 4 years, may be purchased for £6197.30; allowing the purchaser 4 per cent. per annum compound interest on his purchase-money?

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3. A reversionary perpetuity of $120 per annum, was purchased for $2073.60.—How many years hence should this perpetuity commence, allowing 5 per cent per annum compound interest on the purchase-money?

OF LIFE ANNUITIES.

LIFE annuities are estimated by probabilities deduced from the usual period of human life, according to observations made on regular bills of mortality, as in the following table.

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TABLE E. Value of 1*l. or dol. annuity, for a single life.*

Age.	3 *per cent*	4 *per cent.*	5 *per cent*	6 *per cent*
9 *or* 10	19.87	16.88	14.60	12.80
8 *or* 11	19.74	16.79	14.53	12.75
7 *or* 12	19.60	16.64	14.47	12.70
13	19.47	16.60	14.41	12.65
6 *or* 14	19.33	16.50	14.34	12.60
15	19.19	16.41	14.27	12.55
16	19.05	16.31	14.20	12.50
5 *or* 17	18.90	16.21	14.12	12.45
18	18.76	16.10	14.05	12.40
19	18.61	15.99	13.97	12.35
4 *or* 20	18.46	15.89	13.89	12.30
21	18.30	15.78	13.81	12.20
22	18.15	15.67	13.72	12.15
23	17.99	15.55	13.64	12.10
3 *or* 24	17.83	15.43	13.55	12.00
25	17.66	15.31	13.46	11.95
26	17.50	15.19	13.37	11.90
27	17.33	15.04	13.28	11.80
28	17.16	14.94	13.18	11.75
29	16.98	14.81	13.09	11.65
30	16.80	14.68	12.99	11.60
2 *or* 31	16.62	14.54	12.88	11.50
32	16.44	14.41	12.78	11.40
34	16.06	14.12	12.56	11.25
36	15.67	13.82	12.33	11.05
38	15.29	13.52	12.09	10.90
1 *or* 39	15.05	13.36	11.96	10.80
40	14.84	13.20	11.83	10.70
42	14.41	12.85	11.57	10.45
44	13.96	12.50	11.29	10 25
46	13.49	12.13	10 99	10.00
48	13.01	11.74	10.68	9.75
50	12.51	11.34	10.35	9.45
52	12.00	10.92	9.99	9.20
54	11.46	10.47	9.63	8.85
56	10.90	10.01	9.24	8.55
58	10.32	9.52	8.83	8.20
60	9.73	9.01	8.39	7.80
62	9.11	8.48	7.93	7.40
64	8.46	7.92	7.43	6.95
66	7.79	7.33	6.91	6.50
68	7.10	6.75	6.36	6.00
70	6.38	6.06	5.77	5.50
72	5.63	5.38	5.15	4.90
74	4.85	4.66	4.49	4.30
76	4.05	3.91	3.78	3.65
78	3.21	3.11	3.03	2.95
80	2.34	2.28	2.23	2.15

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The numbers in table E, foregoing, may be calculated thus:

Put 86 less any given age $= t$.

R and r, as in the foregoing articles.

$R^t - 1 = m$, to be found in Tab. A.

Then, $Lm - Lt - 2\,Lr - LR \times (t-1) = Lx$; and $\frac{1}{r} - x =$ tabular number for age $86 - t$.

Ex. Suppose the age $= 46$ years, $R = 1.05$, cons. $t = 40$, and m (Tab. A) $= 6.0399887$.

Lm	=	0.781036 +
Lt	=	1.602060
$2\,Lr$	=	3.397940
$LR \times (t-1)$	=	0.826371
Lx	=	0.954665, and $x = 9.0085$
$\frac{1}{r}$	=	20.
Tab. num. for age 46, at 5 per cent.		10.9912

In the following theorems, where one or two lives only are concerned,

Let a, b, = the value or present worth of £1 or $1, annuity, to continue during the life of a person of the age of A, B, respectively, at the time of purchase, found in Table E, opposite the given age, and under the given rate per cent.

r = the interest of £1, or $1, for one year, at any given rate per cent.

s = the present value of $1, &c. annuity to continue during the *single* life of a person whose age is A.

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j = the present value of £1, &c. annuity to continue during the *joint* lives of two persons, whose ages are A, B, respectively.

l = the present value of £1, &c. annuity during the continuance of the *longest* life of two persons, whose ages are A, B, respectively.

v = the present value of £1, &c. annuity to continue during the time that a person of a given age B, may happen to *survive*, another person of a given age A.

Theor. 1. $s = a$.

2. $j = ab \div (a + b - abr)$.

3. $l = a + b - j$.

4. $v = b - j$.

NOTE. The value of £1, &c. annuity, being found as above, is, in all cases, to be multiplied by the given annuity.

EXPLANATORY EXAMPLES.

Suppose $u = 1$, $A = 50$, $B = 40$, $r = .05$, cons. $a = 10.35$, $b = 11.83$; to find s, j, l, and v, respectively.

By theor. 1. $s = 10.35$.

2. $j = 10.35 \times 11.83 \div (10.35 + 11.83 - 10.35, \times 11.83 \times .05) = 7.62$.

3. $l = 10.35 + 83 - 7.62 = 14.56$.

4, $v = 11.83 - 7.62 = 4.21$.

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EXAMPLES FOR PRACTICE, IN LIFE ANNUITIES.

1. What sum should a person of 48 years of age, give for $500 annuity, to continue during his natural life; reckoning interest at 5 per cent ?

2. What is the present worth of a freehold estate (perpetuity) of $400 per annum, but not to commence till after the decease of a person who is now 60 years of age, reckoning interest at 6 per cent. ?

3 Required the present worth of $500 annuity, to continue during the *joint* lives of two persons, one 50, and the other 60 years of age; reckoning interest at 6 per cent.

4. Required the present value of a freehold estate (perpetuity) of $350 per annum; to commence on the decease of two persons, now each 30 years of age; interest at 5 per cent.

5. Required the present worth of an annuity of $700 to continue during the *longest* life of two persons, one of 25, and the other of 18 years of age; interest 6 per cent.

6. Required the present value of $500 annuity, to continue during the *longest* life of two persons, each 25 years of age; interest 5 per cent.

7. Required the present worth of $100 annuity, to continue during the life of a person now 30 years of age; but not to commence till after the decease of a person now 50 years of age; interest 6 per cent.

8. Required the present worth of $200 annuity, to commence on the decease of a child of 5 years of age, and then to continue during the life of a person now 60 years of age, interest 6 per cent.

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OF LIFE ANNUITIES, IN WHICH THREE OR MORE LIVES, A, B, C, &c. ARE CONCERNED.

General Rule.

1. Find, by theorem 2d, last article, the value of £1 annuity, to continue during the joint lives of *A* and *B*, which call x, and consider it as the value or tabular number of the annuity for a single life *X*. 2. Find, in like manner, the value of the annuity during the joint lives of *X* and *C*, which call y, the tabular number for a single life *Y*; and thus proceed finding z, u, &c. with the remaining life annuitants. Then x will be the tabular number or value of £1 annuity during the joint lives of *A* and *B*; y, the value of the annuity during the joint lives of *A*, *B*, *C*; z, the same, during the joint lives of *A*, *B*, *C*, and *D*, &c. &c. In the same manner, by theorem 3d, you may find the value of £1 annuity, during the *longest* life of *A* and *B*; of *A*, *B*, and *C*; of *A*, *B*, *C*, and *D*, &c. and, by theorem 4th, the value of the *reversionary* annuity, during the life of *one*, or of *more joint* survivors, after the termination of the *shortest* or of the *longest life* of any two or more persons, &c.

EXPLANATORY EXAMPLES.

Suppose the ages of four persons, *A*, *B*, *C*, *D*, as follows,

A, 16 years, - - -	Tab. num. 12.5
B, 21 - - - - - - - - - -	12.2
C, 27 - - - - - - - - - -	11.8
D, 40 - - - - - - - - - -	10.7

Interest on the purchase-money, 6 per cent.

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Q. 1. Required the value of £1, annuity during the joint lives of *A* and *B*, (2,) of *A*, *B*, and *C*, (3,) of *A*, *B*, *C*, and *D*.

Q. 2. Required the value of £1, annuity during the *longest* life of *A* and *B*, (2,) of *A*, *B*, and *C*, (3,) of *A*, *B*, *C*, and *D*.

Q. 3. Required the value of £1, reversionary annuity during the *survivorship* of *A*, (1,) after the decease of *B*, (2,) after the decease of *B* and *C*, (3,) after the decease of *B*, *C*, and *D*.

Q. 4. Required the value of £1, *reversionary* annuity during the *joint* survivorship of *A* and *B*, (1,) after the *shortest* life of *C* and *D*, (2,) after the *longest* life of *C* and *D*.

Q. 5. Required the value of £1 reversionary annuity during the *joint* survivorship of *A* and *B*, (1,) after the decease of *C*, (2,) after the decease of *C* and *D*.

From Tab. *B*, $a = 12.50$
$b = 12.20$
$c = 11.80$
$d = 10.70$

By theor. 2. $12.5 \times 12.2 = 152.5 : \times .06 = 9.15$
$12.5 + 12.2 - 9.15 = 15.55) 152.5 (9.80 =$
x, or j, of *A* and *B*.
$11.8 \times 9.8 = 115.64 : \times .06 = 9.64$
$11.8 + 9.8 - 9.64 = 14.66) 115.64 (7.88 =$
y, or j, of *A*, *B*, and *C*.
$10.7 \times 7.88 = 84.316 : \times .06 = 5.06$
$10.7 + 7.88 - 5.06 = 13.52) 84.316 (6.24 =$
z, or j, of *A*, *B*, *C*, and *D*.

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By theor. 3. $12.5+12.2-9.80=14.90=l$, of A and B.
$14.90+11.80-7.88=18.82=l$, of A, B, and C.
$18.82+10.70-6.24=23.28=l$, of A, B, C, and D.

By theor. 4. $12.50-9.80=2.70=v$, of A after life of B.
$12.50-7.88=4.62=v$, of A after *shortest* life of B and C.
$12.50-6.24=6.26=v$, of A after *shortest* life of B, C, and D.

Preparatory

$12.2\times11.8=143.96:\times.06=8.64$
$12.2+11.8-8.64=15.36)143.96($ 24. / 9.37
$14.63=l$, of B & C.
$14.63\times10.7=156.541:\times.06=9.39$
$14.63+10.7-9.39=15.94)156.541($ 25.33 / 9 82
$15.51=l$, of B, C, & D.

$14.63\times12.5=182.875:\times.06=10.97$
$14.63+12.5-10.97=16.13)182.875($ 12.50 / 11.34
$1.16=v$, of A after B and C.

$15.51\times12.5=194.875:\times.06=11.69$
$15.51+12.5-11.69=16.32)194.875($ 12.50 / 11.45
$1.05=v$, of A after B, C, and D.

$11.8\times10.7=126.26:\times.06=8.77$
$11.8+10.7-8.77=13.73)126.26(9.19=j$, of C and D.
$11.8+10.7-9.19=12.31=l$, of C and D.

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$9.80 \times 9.19 = 90.062 : \times .06 = 5.40$

$9.80 + 9.19 - 5.40 = 13.59) 90.062 (6.63$

$3.17 = v$, of A & B, after *shortest* life of C & D.

$9.80 \times 12.31 = 120.638 : \times .06 = 7.24$

$9.80 + 12.31 - 7.24 = 14.87) 120.638 (8.11$

$1.69 = v$, of A and B, after *longest* life of C & D.

EXAMPLES FOR PRACTICE, IN LIFE ANNUITIES, IN WHICH MORE THAN TWO LIVES ARE CONCERNED.

1. Required the present worth of an annuity of $200, to continue during the *joint* lives (*shortest life*) of three persons, A, B, C, of the following ages, respectively; A 18 years, B 25, C 30; reckoning interest at 5 per cent.

2. Required the present worth of an annuity of $50, to continue during the longest life of four persons, A, B, C, D; the age of A being 40 years, B 45, C & D, each 50; interest 6 per cent.

3. Required the present worth of a reversionary annuity of $100, payable to A, now 30 years of age, after the decease of B, 30 years old, and C, 46; interest 5 per cent.

Sundry other cases relating to life annuities, may be imagined; such as, when annuites for a certain term are connected with them, or when leases on life are to be renewed, by "putting in a new life, in place of one that has dropped;" but as such cases can seldom, if ever, occur, in this country, they are here omitted.

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OF PERMUTATION.

PERMUTATION is that operation in arithmetic by which is found the number of *changes* which may be made in the arrangement of any given number of articles; as letters, figures, persons, &c., placed in succession.

Rule.] Take a series of numbers, in arithmetical progression, whose first term, and common difference, is unity, viz. 1, 2, 3, &c., and the last term, or number of terms, equal the given number in the question; then multiply the terms of this series continually together, and the product will be the number of changes, or different arrangements required.

EXPLANATORY EXAMPLES.

1. Required the number of different arrangements in which the figures 1, 2, 3, may be placed, one after another:

$1 \times 2 \times 3 = 6$, Ans—namely, 123, 132, 213, 231, 312, 321.

2. How many different ways may the six vowels, a, e, i, o, u, y, be placed, one after another?

$1 \times 2 \times 3 \times 4 \times 5 \times 6 = 720$, Ans.

EXAMPLES FOR PRACTICE, IN PERMUTATION.

1. The bells in Christ's Church belfry, Philadelphia, are eight in number. How many different changes may be rung on these bells?

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2. How many different changes may be made in the order of arrangement of the words in the following famous Latin verse, viz.:

*Tot tibi sunt dotes, Virgo, quot sidera cælo**.

OF COMBINATIONS.

THIS operation teaches how to find the number of all the possible combinations that can be formed out of any given number of particulars, subject to certain conditions, as in the following problems:

Prob. I. To find how many combinations, each consisting of a given number of particulars, may be formed out of any given number; no one particular being *repeated* in any combination, nor all the particulars in one, the same with those in another, though differently arranged.

Rule.] Take two series of numbers, in arithmetical progression, both having unity for their common difference, the first *increasing* from unity to the number composing one of the combinations; and the second, consisting of the same number of terms, but *decreasing*, from the given number out of which the combinations are to be formed. Multiply the terms of the latter series

* The latinist will readily perceive, that all the sentences thus formed will be strictly grammatical, and some thousands of them pure hexameter verses.

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continually together, for a divldend, and those of the former series, in like manner, for a divisor; and then the quotient of this division, (taking aliquot parts of multiplying and dividing terms, as usual,) will be the number of combinations required.

NOTE. The number of *all the possible combinations*, of twos, threes, &c. which can be formed, subject to the above conditions, may be found thus:—Raise the number *two* to a power whose index shall equal the given number out of which the combinations are to be formed, from which power subtract the said number, *more unity*, and the remainder will be the number of the combinations required.

EXPLANATORY EXAMPLES.

1. How many combinations, each consisting of two different letters, can be formed out of the three letters, *a*, *b*, *c*?

$$\frac{3 \times 2}{1 \times 2} = 3.$$ Ans.—namely, *ab*, *ac*, *bc*.

2. How many combinations, each consisting of three different figures, may be formed out of the seven figures, 1, 2, 3, 4, 5, 6, 7?

$$\frac{7 \times 6 \times 5}{1 \times 2 \times 3} = 35.$$ Ans.

3. How many *possible* combinations, each consisting of two or more different particulars, may be formed out of seven articles?

$$2^7 - 8 = 120.$$ Ans.

Prob. II. To find how many combinations, each consisting of a given number of particulars, *no one being repeated* in the same combination; but a different arrange-

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ment, being considered as a different combination, may be formed out of any given number.

Rule.] Take a series of numbers in arithmetical progression, beginning with the number out of which the combinations are to be formed, and decreasing by unity, till the number of terms shall equal the number of particulars in each combination; then, the terms of this series being continually multiplied together, will give the number of combinations required.

NOTE. To find the number of all the *possible* combinations, of two or more particulars, that can be formed out of any given number, subject to the above conditions, you must find the number, as above, in each particular case, and then take their sum.

EXPLANATORY EXAMPLES.

1. How many numbers, each consisting of two different figures, can be formed out of the figures, 1, 2, 3?

$3 \times 2 = 6$. Ans.—namely, 12, 13, 23, 21, 31, 32.

2. How many numbers, each consisting of three different figures, can be formed out of the nine digits?

$9 \times 8 \times 7 = 504$. Ans.

3. Required the number of all the *possible* combinations, of two or more different figures, that can be formed out of the nine digits.

$9 \times 8 = 72$
$\times 7 = 504$
$\times 6 = 3024$
$\times 5 = 15120$
$\times 4 = 60480$
$\times 3 = 181440$
$\times 2 = 362880$
$\times 1 = 362880$

Sum = 986400 Ans.

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Prob. III. To find how many combinations, each consisting of a given number of particulars, in which one or more may be repeated, and a different arrangement considered as a different combination, can be formed out of any given number.

Rule.] Raise the given number out of which the combinations are to be formed, to a power whose index shall equal the number contained in each combination, and this power will be the number of combinations required.

NOTE. To find, in this case, the number of all the *possible* combinations, of two or more particulars—Raise the given number out of which the combinations are to be formed, to a power whose index shall equal said number, from which subtract unity: divide the remainder by the said number less unity, multiply the quotient by said number, and from the product subtract the said number, and then the remainder will be the number required.

EXPLANATORY EXAMPLES.

1. How many numbers, each consisting of two figures, may be formed out of the three figures, 1, 2, 3?

$3^2 = 9$. Ans.—namely, 12, 13, 23, 21, 31, 32, 11, 22, 33.

2. How many numbers, each consisting of three figures, may be formed out of the nine digits?

$9^3 = 729$. Ans.

3. How many *possible* numbers, consisting of two or more places, may be formed out of the five figures, 1, 2, 3, 4, 5?

$5^5 - 1 = 3124 : \div 4 = 781 : \times 5 = 3905 : - 5 = 3900$. Ans.

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Prob. IV. To find how many combinations, each consisting of a given number of particulars, admitting a repetition of particulars in the same combination, but no permutation or change in the arrangement, may be formed out of any given number.

Rule.] Find the numbers of such combinations, subject to the conditions in problems II. & III. respectively; and the difference of these numbers will be the number of combinations required.

NOTE. The number of *all* the *possible* combinations in this case, may be found by the same rule.

EXPLANATORY EXAMPLES.

1. How many combinations, each consisting of three particulars, subject to the above conditions, may be formed out of four different quantities?

$4 \times 3 \times 2 =$ - - 24 by prob. II.
$4^3 =$ - - - - - 64 ——— III.

Diff. = 40 Ans.

2. How many *possible* combinations, subject to the conditions in this problem, may be formed out of four different quantities?

$4 \times 3 =$ - - - - - 12
- - - $\times 2 =$ - - - - 24
- - - - - - $\times 1 =$ - - 24

Sum = 60, by prob. II.
$(4^4 - 1) \times \frac{4}{3} - 4 =$ - - - 336, ——— III.

Diff. = 276 Ans.

Prob. V. To find how many combinations may be formed, each containing a given number of particulars,

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taken respectively from two or more ranks, containing severally, a given number of different quantities; these combinations being further subject to one or other of the conditions in the foregoing problems.

Rule.] Find, according to the specified conditions, the number of combinations contained in each of the given ranks, separately, and let these numbers be multiplied continually together; then, if the required conditions be those in Problems I. or IV., this product will be the answer. But if the conditions be those in Problems II. or III., then, multiply the above product, by the number of permutations or changes which may be made out of the whole number contained in one of the combinations formed from the several ranks, and the last product will be the answer required.

NOTE. The number of *all the possible* combinations, in this case, may be found, by a similar process.

EXPLANATION EXAMPLES.

1. Suppose each of the required combinations to contain

3 particulars out of a rank of 5
2 - - - - - - - - - - 3
1 - - - - - - - - - - 2

Case 1, Subject to the conditions in Prob. I.

$$\left.\begin{array}{l} \frac{5\times4\times3}{1\times2\times3} = \text{- - } 10 \\ \frac{3\times2}{1\times2} = \text{- - - - } 3 \\ \frac{2}{1} = \text{- - - - - } 2 \end{array}\right\} \text{Prod.} = 60, \text{ Ans.}$$

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Case 2. Subject to the conditions in Prob. IV.

$5^3 =$ - - 125 } diff. = 65 }
$5 \times 4 \times 3 =$ 60 } } Prod. = 390 - - - Ans.
$3^2 =$ - - - 9 } diff. = 3 }
$3 \times 2 =$ - - 6 }
1 in 2 = 2 }

Case 3. Subject to the conditions in Prob. II.

$5 \times 4 \times 3 =$ - 60 }
$3 \times 2 =$ - - 6 } Prod. = 720 }
$2 \times 1 =$ - - 2 } } Prod. = 518400 - - Ans.
$6 \times 5 \times 4 \times 3 \times 2 \times 1$ - - - 720 }

Case 4. Subject to the conditions in Prob. III.

$5^3 =$ - - 125 }
$3^2 =$ - - 9 } Prod. = 2250 }
$2^1 =$ - - 2 } } Prod. = 1620000 - - Ans.
$6 \times 5 \times 4 \times 3 \times 2 \times 1 =$ - - 720 }

2. Required the number of *all the possible* combinations, that can be formed out of the above given numbers, subject to the conditions in Prob. I.

$2^5 - 6 = 26$ }
$2^3 - 4 = 4$ } Prod. = 104 - - Ans.
$2^2 - 3 = 1$ }

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EXAMPLES FOR PRACTICE, IN COMBINATIONS.

In Prob. I.

1. How many numbers, each consisting of three different figures, of which the first is the least, and the third the greatest, may be formed out of the nine digits?

In Prob. II.

2. How many numbers, each consisting of seven different figures, may be formed out of the nine digits?

In Prob. III.

3. How many numbers, each consisting of seven places, may be formed out of the nine digits?

In Prob. IV.

4. How many combinations each consisting of five letters, (admitting a repetition but no permutation,) may be formed out of ten different letters?

In Prob. V.

5. How many different ways may a file of men be taken from three ranks; the first rank consisting of 12 men, the second rank of 9, and the third rank 7? a file being one man from each rank.

PROMISCUOUS.

6. How many chimes, each consisting of three different musical notes, may be rung on an octave of bells?

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7. How many syllables, each consisting of a vowel and a consonant, admitting permutation, may be formed out of the five vowels, a, e, i, o, u, and the seven consonants, b, d, f, l, m, n, r?

8. How many combinations of two or more letters, one at least of each being a vowel, may be formed out of 5 vowels and 6 consonants? repetition and permutation being admitted.

OF THE DOCTRINE AND ARITHMETIC OF CHANCES.

When the occurrence of any future event cannot be foretold by any human knowledge, or calculation, it is, in common language, said to be a *contingency*, or to happen *by chance*. Chance, however, is not to be considered as an efficient cause of any event whatever; being no more than a conception of the mind, and, in fact, a perfect nonentity. But though the occurrence of such an event cannot be foretold with *certainty*, yet the degree of its *probability*, under given circumstances, is a subject of arithmetical calculation.

These calculations, though frequently employed in games of hazard, are by no means confined to this subject; but are applicable also to problems in life annuities, in finding the most probable mean result of any given number of astronomical or other observations, &c.

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The application of the doctrine of chances may be considered as nearly inexhaustible; yet the following problem will, I presume, comprehend most of the cases which usually occur on this subject.

PROBLEM.

To find the degree of probability of the occurrence of any contingent future event, subject to any given conditions.

SOLUTION.

From the nature of the question, consider and determine the whole number of *equally probable* events, whether simple or combined, of which the event in question is one; also, the number of *ways* in which, according to the conditions of the question, this event may, with equal probability, take place; then, the fraction, of which the latter number is the numerator, and the former number the denominator, will express the degree of probability required: *certainty* being considered as the unit.

Questions in chance are sometimes limited to a certain event *only*, or exclusive of others; sometimes to a certain event *at least*, or inclusive of others; and sometimes to one *or other* of two or more events.

The doctrine of chances is most generally, and very conveniently, illustrated by the casting of dice, the twirling of a totum, or the drawing of lots*.

* The use of the above terms, which could not be well avoided, it is hoped, will give no countenance to the immoral practice of *gaming*, where they are also very frequently employed.

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EXPLANATORY EXAMPLES.

Ex. 1. Suppose seven balls, viz. three white and four black, to be promiscuously thrown into a bag or urn, and one of them drawn out at random.

(1.) Required the chance that this will be a white ball.

Here, the *whole* number of equally probable events will be 7; and the number of *ways* in which the event in question may happen will be 3; hence $\frac{3}{7}$ expresses the chance required.

(2.) Required the chance that the single ball drawn out will be a black one.

Here, as above, the whole number of chances will be 7; and the number of ways in which the required event may take place, will be 4; hence $\frac{4}{7}$ expresses the chance required. Now, since the ball drawn out must necessarily be either a white one or a black one, it follows, that the two above fractions being added together will equal unity, or certainty. Hence, the fraction expressing the chance of any event taking place, being subtracted from unity, will give the fraction expressing the chance of its failure, or not taking place; and vice versa.

Ex. 2. Suppose, as in the above example, and that two balls be drawn out together:

(1.) Required the chance that these will be both white balls.

Here, the whole number of chances, drawing out two at once, will be the combinations of 2 in $7 = \frac{7 \times 6}{1 \times 2} = 21$,

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(see Combinations, case 1,) and the number of ways of drawing out two white balls, will be the combinations of 2 in $3 = \frac{3\times2}{1\times2} = 3$: Hence, $\frac{3}{21} = \frac{1}{7}$ expresses the chance required.

(2.) Required the chance that the two balls drawn out will be both black ones.

Here, the whole number of chances will, as before, be 21, and the number of ways of drawing out two black balls, will be the combinations of 2 in $4 = \frac{4\times3}{1\times2} = 6$: Hence, $\frac{6}{21} = \frac{2}{7}$ expresses the chance required.

(3.) Required the chance that the two balls drawn out will be, one white and one black.

Here, as before, the whole number of chances will be 21, and the number of ways in which a white and a black may be drawn out, will be the combinations of 1 in 3, (3) × the combinations of 1 in 4 (4) = 12: Hence, $\frac{12}{21} = \frac{4}{7}$ expresses the chance required. Now, the sum of the above three fractions, viz. $\frac{1}{7} + \frac{2}{7} + \frac{4}{7}$ = unity, or a certainty, as is evident from the conditions of the question.

The three cases in *Ex.* 2, and all other similar cases, may, in a more simple manner, be solved, by supposing a single ball to be drawn *twice* in succession, and then taking the product of the two resulting fractions—Thus, (in case 1,) the chance of drawing a white ball, the first trial, will be $\frac{3}{7}$, and that of drawing a white ball, the second trial, (one white ball being drawn,) will be $\frac{2}{6}$: Hence $\frac{3}{7}\times\frac{2}{6} = \frac{1}{7}$, the chance required. Again, (in case 2,) the chance in favour of a black ball, the first trial, will be $\frac{4}{7}$, and the second trial, $\frac{3}{6}$: Hence $\frac{4}{7}\times\frac{3}{6} = \frac{2}{7}$, the

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chance required. Lastly, the chance in favour of a white ball, the first trial, will be $\frac{3}{7}$, and that in favour of a black ball, the second trial, will be $\frac{4}{6}$: Hence $\frac{3}{7}\times\frac{4}{6}=\frac{2}{7}$, the chance of a white ball the first, and a black ball the second. But the conditions will be equally answered by drawing a black ball the first, and a white ball the second; the chance of which will be $\frac{4}{7}\times\frac{3}{6}$, also $=\frac{2}{7}$; and since the chance of the one *or* the other is evidently = the sum of the two, it follows, that $\frac{2}{7}+\frac{2}{7}=\frac{4}{7}$, will be the chance required.

Ex. 3. Suppose a bag, or urn, to contain 12 balls, viz. 3 white, 4 black, and 5 red; and that three of these are drawn at random, either all together, or one after another, in succession.—Required the chance,

(1) That they shall be *all* white balls.

Here, (considering them to be drawn in succession,) the chance in favour of the *first* being a white ball, will be $\frac{3}{12}$, that in favour of the second $\frac{2}{11}$, and that in favour of the third $\frac{1}{10}$; the product of which fractions $=\frac{1}{220}$, the chance required.

In the same manner it may be found, that the chance in favour of them being *all* black balls, will be $\frac{4}{12}\times\frac{3}{11}\times\frac{2}{10}=\frac{1}{55}$; and that in favour of them being *all* red, will be $\frac{5}{12}\times\frac{4}{11}\times\frac{3}{10}=\frac{1}{22}$.

(2.) That two *only* shall be white.

Here, (considering them as all drawn together,) the *whole* number of chances will be = the combinations of 3 in 12 $=\frac{12\times11\times10}{1\times2\times3}=220$, of which the chances in favour of the event in question, will be the = combinations of 2 in 3 = 3: Hence, $\frac{3}{220}$ will express the chance required.

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In the same manner it may be found, that the chance in favour of two *black* balls only, will be $\frac{6}{220} = \frac{3}{110}$, and that in favour of two red, $\frac{10}{220} = \frac{1}{22}$.

(3.) That two, *at least*, shall be white; i. e. two *or* three.

Here, the chance in favour of two only $= \frac{3}{220}$, and that in favour of three $= \frac{1}{220}$; the *sum* of which fractions $= \frac{4}{220} = \frac{1}{55}$, will be the chance required.

In the same manner it may be found, that the chance in favour of two, *at least*, being black, will $= \frac{6+4}{220} = \frac{1}{22}$, and that in favour of two, *at least*, being red, will $= \frac{10+10}{220} = \frac{1}{11}$.

(4.) That one *only* shall be white and another black; i. e., a white, a black, and a red.

Here, the chances in favour of the event in question, will $= \frac{3}{11} \times \frac{4}{11} \times \frac{5}{10} = \frac{1}{22}$, &c. &c.

Ex. 4. Suppose three bags, *A*, *B*, *C*; *A* containing 3 white balls and 2 black, *B* containing 4 white and 5 black, and *C* containing 5 white and 3 black. Required the chance, that in drawing out a single ball from one of these bags, indiscriminately, it shall be a white one.

Here, the chance in favour of drawing the ball from the bag *A*, will be $\frac{1}{3}$, and that of drawing a white ball from said bag, will be $\frac{3}{5}$: Hence, $\frac{1}{3} \times \frac{3}{5} = \frac{1}{5}$, will be the chance with bag *A*. For the same reason, the chance with bag *B* will be $\frac{1}{3} \times \frac{4}{9} = \frac{4}{27}$; and that with bag *C*, $\frac{1}{3} \times \frac{5}{8} = \frac{5}{24}$. Now the sum of these three fractions, viz. $\frac{1}{5} + \frac{4}{27} + \frac{5}{24} = \frac{601}{1080}$ will be the chance required.

In the same manner may be found the chances in a variety of other cases; for instance, that in drawing out

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two balls, they shall be both white, or both black, or one white and the other black, &c. &c.

Ex. 5. Required the chance, that in picking up the seven types, *b, e, i, l, r, t, y*, one by one, thrown promiscuously into one of the cells of a printer's case, and setting them in the stick, the word *liberty* shall be composed.

Here, the chance in favour of the first letter (*l*,) will evidently be $\frac{1}{7}$, and that in favour of the remaining letters, exclusive of the last, which will be a certainty, will be $\frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}$; consequently the product of these six fractions $= \frac{1}{1\,0\,4\,0}$, will be the chance required.

Ex. 6. Required the probability, or chance, that a person of 20 years of age will live to the age of 30.

The following Table, taken from M. Parcieux, exhibits the state of mortality in France; the first line containing the different ages from birth, to 90 years; the second containing the number of persons alive, of the respective ages, out of 1000 born at the same time; and the third line, the *difference*, or persons dying in 10 years.

Ages——	0,	10,	20,	30,	40,	50,	60,	70,	80, 90.
Living.—	1000,	660,	610,	550,	492,	436,	349,	232,	88, 8.
Dying.—	340,	50,	60,	58,	56,	87,	117,	144,	80

From the above Table, it appears that $\frac{5\,5\,0}{6\,1\,0} = \frac{5\,5}{6\,1}$ expresses the chance of living 10 years, and consequently $1 - \frac{5\,5}{6\,1} = \frac{6}{6\,1}$ or $\frac{1}{1\,0}$ nearly, the chance of dying.

Ex. 7. Required the probability that two persons, *A* and *B*, one aged 30 and the other 40, will both live 10 years.

The chance in favour of $A = \frac{4\,9\,2}{5\,5\,0}$, $(\frac{3\,4\,6}{3\,7\,7},)$ and that in

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favour of $B = \frac{486}{492}$ ($\frac{109}{123}$,) the product of which fractions $= \frac{26814}{33825}$, the chance required.

Consequently, $1 - \frac{26814}{33825} = \frac{7011}{33825}$, will be the chance of one, *at least*, dying; $1 - \frac{246}{275} \times 1 - \frac{109}{123} = \frac{406}{33825}$, will be the chance of *both* dying; also, $1 - \frac{109}{123} \times \frac{246}{275} = \frac{3444}{33825}$, will be the chance of *B only* dying, and $1 - \frac{246}{275} \times \frac{109}{123} = \frac{2164}{33825}$, will be the chance of *A only* dying.

Ex. 8. Suppose a single die, having its six plane sides numbered 1, 2, 3, &c., respectively.

(1.) Required the chance, that, in a single throw, the *ace*, (side No. 1.) or any other determinate side, shall be turned up.

Here, the whole number of chances will be 6, the number of sides on the die, and 1 of these only the side in question: Hence, $\frac{1}{6}$ expresses the chance required.

(2.) Supposing as above, required the chance that any side, *except* the ace, shall be turned up.

Here, the whole number of chances will be 6, as before; and the number of ways, in which the event in question may happen, will be 5: Hence, $\frac{5}{6}$ expresses the chance required.

In this example, to turn up an ace, or *no* ace, must be a certainty, and, accordingly, the sum of the two chances $\frac{1}{6} + \frac{5}{6} = 1$.

Ex. 9. Suppose two dice, numbered as above.

(1.) Required the chance that, in a single throw, the two aces, (or any other two given sides) will be turned up.

Here, the whole number of chances will be 36, viz. the 6 faces on one die combined with the 6 faces on the other; and of these, one only will be in favour of the event in question: Hence, $\frac{1}{36}$ expresses the chance required.

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(2.) Required the chance that, in a single throw, one ace, and no more, will be turned up.

Here, the whole number of chances will, as before, be $6\times6=36$; and of these, 10 will be in favour of the event in question, viz. the ace on each die, combined with the remaining 5 sides, on the other: Hence, $\frac{10}{36}=\frac{5}{18}$, expresses the chance required.

(3.) Supposing as above, required the chance that one ace, *at least*, i. e. one or two, shall be turned up.

Here, as before, the whole number of chances will be 36, of which those in favour of *one* ace *only*, will be 10, as before, and the chance in favour of two aces will be only *one*: Hence, the sum $\frac{11}{36}$ will be the chance required.

Ex. 10. Suppose *three* dice, numbered as above.

(1.) Required the chance that, in a single throw, the three aces, (or any other given triplet, three equal numbers, or any other three particular sides,) will be turned up.

Here, the whole number of chances will be $6\times6\times6=216$, of which *one only* is in favour of the event: Hence, $\frac{1}{216}$, expresses the chance required.

(2.) Required the chance that, in a single throw, *two* aces *only* will be turned up.

Here, as above, the whole number of events will be 216, of which 15 will be favourable to the event, viz. the combinations of 2 aces in 3, $(=3)\times5$; the sides on the third die, exclusive of the ace: Hence, $\frac{15}{216}=\frac{5}{72}$, expresses the chance required.

(3.) Required the chance that, in a single throw, *one* ace *only* will be turned up.

Here, as above, the whole number of chances will be

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216, of which 75 will be in favour of the event, viz. the combinations of one in three, $(=3) \times 25$; the combinations of 5 with 5: Hence, $\frac{75}{216} = \frac{25}{72}$, expresses the chance required.

(4.) Supposing as above, required the chance that *two* aces, *at least*, i. e. either two or three, will be turned up.

Here, the chance required will equal the sum of the chances for *two only*, and *three only* = (as above) $\frac{15}{216} + \frac{1}{216} = \frac{2}{27}$, the chance required.

(5.) Supposing as above, required the chance of turning up a triplet of any kind, without distinction.

Here, as before, the whole number of chances will be 216; and of these, there will be 6 triplets: Hence, $\frac{6}{216} = \frac{1}{36}$, expresses the chance required.

(6.) Supposing as above, required the chance of turning up a doublet, (two equal numbers) of any kind, without distinction.

Suppose the three diee marked A, B, C, respectively; then the doublets on A & B, which would be 6 in number, might be combined with the faces on the die C, 30 different ways, without forming a triplet, viz.

$$\left.\begin{matrix} A\,1 \\ B\,1 \end{matrix}\right\} \overbrace{2, 3, 4, 5, 6,}^{C} \quad \left.\begin{matrix} A\,2 \\ B\,2 \end{matrix}\right\} \overbrace{1, 3, 4, 5, 6,}^{C} \text{ \&c.}$$

And the same number of combinations might be formed with the 6 doublets, on A and C, with the third B; also with those on B and C, and with the third A: Hence, $\frac{30+30+30}{216} = \frac{5}{12}$, will be the chance required.

In the same manner it may be found, that the chance

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of turning up a doublet with a single throw of *four* dice, would be $\frac{900}{1296} = \frac{25}{36}$; and of turning up *a triplet*, would be $\frac{240}{1296} = \frac{5}{324}$, &c.

Ex. 11. Required the chance that, in a single throw, with two dice, the *sum* of the numbers turned up on the two faces, shall equal 7, and no more.

Here, the whole number of chances will be 36, of which would give the number in question, viz. 1 and 6, 2 and 6, 3 and 4, 4 and 3, 5 and 2, 6 and 1; Hence, $\frac{6}{36} = \frac{1}{6}$, will express the chance required.

In the same manner it may be found, that the chance in favour of 6 or 8, and no more, will be $\frac{5}{36}$; in favour of 5 or 9, and no more, $\frac{4}{36} = \frac{1}{9}$; &c.

Ex. 12. Required the chance that, in a single throw, with two dice, the sum of the two numbers turned up shall be 7, *at least*; i. e. 7, 8, 9, 10, 11, or 12.

Here, as in other similar cases, the chance required will equal the sum of the individual chances, separately considered; in this case, $= \frac{6+5+4+3+2+1}{36} = \frac{21}{36} = \frac{7}{12}$.

NOTE. Most of the cases which usually occur, in the casting of dice, or in other chances of a similar nature, may be solved with the utmost facility, by the following *formulæ*, being the different powers of the binomial (quantity of two terms) $a + b$; in which, so far as respects common dice, $a = 1$, and $b = 5$.

1st power, $\left.\begin{matrix} a + b \\ 1 + 5 \end{matrix}\right\} = 6.$

2d ——— $\left.\begin{matrix} a^2 + 2ab + b^2 \\ 1 + 10 + 25 \end{matrix}\right\} = 36.$

3d ——— $\left.\begin{matrix} a^3 + 3a^2b + 3ab^2 + b^3 \\ 1 + 15 + 75 + 125 \end{matrix}\right\} = 216.$

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4th power, $a^4+4a^3b+6a^2b^2+4ab^3+b^4$
$1+20+150+500+625 \Big\} = 1296.$

5th Power.

$a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$
$1+25+250+1250+6125+6125 \Big\} = 7776.$

6th Power.

$a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$
$1+30+375+2500+9375+36750+30625 \Big\} =$
46656. &c. &c.

EXPLANATION AND APPLICATION.

When the chance is to be determined by a single cast of any number of dice, (which is equivalent to an equal number of casts with a single die,) then take that power of the binomial whose index equals the number of dice; and the term in this binomial in which the index of a equals the number of faces in the question to be turned up, will be the numerator, and the whole power of the binomial will be the denominator of the fraction expressing the chance required.

Ex. 1. Required the chance of turning up an ace, (or any other determinate face) in a single throw with *one* die.

Here, the answer will be found in the *first* power, in which the term $a=1$, and the whole power $= 6$: Hence, $\frac{1}{6}$ expresses the chance required.

Ex. 2. Required the chance of turning up *one* ace *only*, in a single throw, with *two* dice.

Here, in the 2d power, the term containing a $(2ab)=10$, and the whole power $= 36$: Hence, $\frac{10}{36}=\frac{5}{18}$, expresses the chance required.

In the same manner it may be found that $\frac{75}{216}=\frac{25}{72}$ is the chance with *three* dice; $\frac{500}{1296}=\frac{125}{324}$, with *four* dice; $\frac{1125}{7776}$, with *five* dice, &c.

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Ex. 3. Required the chance of turning up two aces, (or any other two faces,) in a single throw, with two dice.

Here, in the 2d power, the term $a^2 = 1$, and the whole power $= 36$: Hence, $\frac{1}{36}$ will be the chance required.

In like manner it will appear that, with *three* dice, the chance of turning up two aces *only*, will be $\frac{15}{216} = \frac{5}{72}$; with *four* dice, $\frac{150}{1296} = \frac{25}{216}$; with *five* dice, $\frac{1250}{7776} = \frac{625}{3888}$, &c.

Ex. 4. Required the chance of turning up *either* an ace *or* a deuce, (or any other two specified faces,) in a single throw, with *two* dice.

Here, the chance required will be equal to the *sum* of the chances of turning up an ace *and* a deuce, separately $= \frac{10+10}{36} = \frac{5}{9}$.

In the same manner it may be found that, with *three* dice the chance would be $\frac{75+75}{216} = \frac{25}{36}$; with *four* dice, $\frac{500+500}{1296} = \frac{250}{324}$, &c.

Ex. 5. Required the chance of turning up *two* aces, *at least*, i. e. two *or* three in a single throw, with *three* dice.

Here, as in *Ex.* 4, the chance will be the *sum* of the two, found separately, $= \frac{15+1}{216} = \frac{2}{27}$. For the same reason the same chance with *four* dice, would be $\frac{150+20+1}{1290} = \frac{57}{432}$; with *five*, $\frac{1250+250+25+1}{7776} = \frac{763}{3888}$, &c.

Ex. 6. Required the chance of turning up any three faces, *except* the three aces, (or any other three specified faces,) in a single throw, with three dice.

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Here, the chance in favour of three aces *only*, would be $\frac{a^3}{216} = \frac{1}{216}$, and this subtracted from unity, will leave $\frac{215}{216}$, the chance required.

The above formulæ may also be applied in determinating the chance in favour of turning up a *doublet*, *triplet*, &c. observing, in this case, that the chance in favour of any specified *two*, *three*, &c. faces must be multiplied by *six*.

Ex. 7. Required the chance of turning up any doublet, indiscriminately, in one throw, with three dice.

Here, in the 3d power, the term containing a^2 $(3a^2b)$ is 15: Hence, $\frac{15\times6}{216} = \frac{5}{12}$ will be the chance required.

Ex. 8. Required the chance of turning up *one* doublet, and *one* triplet, *only*, in one throw, with *four* dice.

Here, in the 4th power, the term containing a^2 $(6a^2b^2) = 150$, and that containing a^3 $(4a^3b) = 20$: Hence, $\frac{150\times6\times20\times6}{1296\times1296} = \frac{125}{1954}$, will be the chance required.

If the question had been to find the chance of turning up either *one* doublet *or one* triplet *only*; then, $\frac{150\times6+20\times6}{1296} = \frac{255}{324}$ would have been the chance required.

And if the question had been to turn up neither a doublet nor a triplet; then, $1 - \frac{255}{324} = \frac{69}{324} = \frac{23}{108}$ would be the chance required.

Cases may occur where it may be required to determine the *number of trials* in which there would be an *equal chance* of success and failure, in any specified event. Such cases, when dice are employed, may also

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be readily solved by the above formulæ of binomial powers, carried, if necessary, to a greater extent. For instance, suppose it were required to determine the *number of throws* in which there would be an *equal chance* of turning up one ace *only*, with a single die.—Or, which is the same thing, to determine the *number of dice*, with which there would be an equal chance of turning up one ace only in a *single* throw. In this case, it will appear, that, if the number of dice (power of the binomial) be four, the chance of success will be $4ab^3 : (a+b)^4$; or, 500: 1296; which is the *nearest* to one-half, though *less*—and there can be no *fraction* of a trial, or number of dice.

If the question had been the same as the above, only to turn up one ace, *at least*; then, with four dice, (4th power) the chance of success would be $1+20+150+500$ (671); 1296, still nearest to one-half, though greater: And, in this manner, may any question of the kind be easily solved.

EXAMPLES FOR PRACTICE, IN THE ARITHMETIC OF CHANCES.

1. What would be the difference in favour of the chance of turning up an ace, in one throw, with a single die, and of turning up two aces in one throw, with two dice?

2. Suppose 12 balls, viz. 7 white, 3 black, and 2 red, to be thrown promiscuously into a bag, and two of them to be drawn out at random. What is the chance that these shall be the two red balls?

3. Suppose three persons, each of the age of 40 years. What is the chance that one *only* of these shall arrive at the age of 50 years?

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APPLICATION OF ARITHMETIC, IN THE SOLUTION OF A FEW SELECT PROBLEMS IN MATHEMATICS.

DEFINITIONS IN TRIGONOMETRY AND MENSURATION OF SUPERFICIES.

1. MAGNITUDES are of three kinds, viz. lines, superfices, and solids. *Lines* have length only; *superficies* have length and breadth; solids have length, breadth, and thickness.

2. Lines are of two kinds, viz. right, and curved. A *right* line is the nearest distance between its extreme points; a *curve* line is not the nearest distance.

3. Lines, with respect to their position to each other, are either parallel, converging, or diverging. *Parallel* right lines are those which, lying in the same plane, are in all parts equidistant from each other, or which, if produced either way, would never meet. *Converging* right lines are those which approach nearer and nearer to each other; or which, if produced, would at length meet. *Diverging* right lines are those which recede farther and farther from each other; or which converge in the opposite direction.

4. A *plane angle* is the inclination or opening of two lines, called the legs or limbs, diverging from the same point, called the angular point. Angles are either right or oblique. When a right line meets or crosses another in such a manner as to make the two adjacent angles equal to each other, these are called *right* angles, and the two lines are said to be *perpendicular*, or at *right angles* to each other. An *oblique* angle is that which is

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either greater or less than a right angle ; if greater, it is called an *obtuse* angle, and if less, an *acute* one.

5. Superficies are of two kinds, viz. plane and curved. A *plane* superficies is perfectly flat, or lies evenly between its boundary lines; as the surface of a table. A *curved* superficies rises above, or sinks below its boundary lines; the former is called a *convex* superficies, as the surface of an egg; the latter is called a *concave* superficies, as the inner surface of a bowl.

6. A *figure* is any bounded space. A *plane* figure, that which is drawn on a plane surface.

7. Plane right-lined figures are particularly denominated from the number of their sides or angles ; viz. one of three sides or angles is called a *trilateral* figure or *triangle ;* one of four, a *quadrilateral* figure, or *quadrangle ;* and one of more than four, a *multilateral* figure, or *polygon.*

8. Triangles, with respect to their sides, are either *equilateral*, having all their sides equal, *isosceles*, having only two sides equal, or *scalene*, having three unequal sides. With respect to their angles, they are either *right-angled*, having one right angle; obtuse-angled, having one obtuse angle ; or *acute-angled*, having all the angles acute. The two latter are also called *oblique* triangles. In a right-angled triangle, the side opposite to the right angle is called the *hypothenuse*, and those including the right angle are called legs, of which one may be called the *base*, and the other the *perpendicular.*

9. Of quadrilateral figures, that is called a *parallelogram* whose opposite sides are respectively parallel; that a *trapezium*, whose opposite sides are not parallel; and that a *trapezoid*, two of whose sides only are parallel.

10. Of parallelograms, that is called a *square*, which is equal-sided and right-angled ; that a *rectangle*, which is right-angled, but not equal-sided ; that a *rhombus*, which is equal-sided, but not right-angled ; and that a *rhomboid*, which is neither equal-sided nor right-angled. A right line joining two opposite angles of any quadrilateral figure is called a *diagonal.*

11. Polygons are particularly denominated from the number of their angles. Thus one of five angles (and

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consequently five sides) is called a *pentagon*; one of six, a *hexagon*; one of seven, a *heptagon*; one of eight, an *octagon*, &c.

12. A *circle* is a plane figure, bounded by a curve line called the *circumference*, which is in all parts equidistant from a certain point within the same called the *center*. A *degree* is the 360th part of the whole circumference; a *minute*, the 60th part of a degree, and a *second*, the 60th part of a minute.

13. Any part of the circumference of a circle is called an *arch*, and the right line joining its extremities, the *chord* of that arch.

14. A chord divides the circle into two parts called *segments*. That chord which passes through the center is called a diameter; and the two segments into which it divides the circle are called *semicircles*.

15. Any right line drawn from the center of a circle to the circumference, is called a *radius* or *semidiameter*.

16. A *sector* is a part of the circle contained under two radii, and the arch-line lying between them; this arch-line being the *measure*, in degrees and parts, of the angle comprehended between the two radii: thus a right angle is one of 90 degrees; an obtuse angle, one greater than 90, and an acute angle, one less than 90. The *complement* of an arch or of an angle, is the quantity it wants of 90 degrees, or of a right angle; and the *supplement*, the quantity it wants of 180 degrees or of two right angles.

17. Sectors are particularly denominated from the proportional part of the circle they contain; thus, a *quadrant* contains a 4th part of the circle; a *quintant*, the 5th part; a *sextant*, the 6th part; an *octant*, the 8th part; &c.

18. An *ellipsis* is an oblong curvilineal figure, resembling a circle: its length being termed the *transverse*, and its breadth, the *conjugate* diameter: and the rectangle of these two diameters equals the square of a *mean* diameter, or that of an equal circle.

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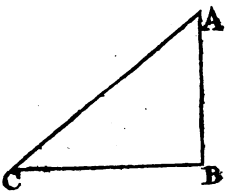
SECT. I. IN TRIGONOMETRY.

Prob. I. Any two sides of a right-angled triangle being given, to find the third side.

Theor. 1. In every right-angled triangle, the square of the *hypothenuse*, is equal to the sum of the squares of the two *legs*.

Ex. 1. In the right-angled triangle ABC, suppose the leg BC = 4 (perches or any other measure) and the leg AB = 3; required the hypothenuse AC.



OPERATION.

$$4 \times 4 = 16$$
$$3 \times 3 = \underline{\ \ 9}$$
$$\sqrt{25} = 5, \text{ the hyp. required.}$$

Ex. 2. Suppose the leg BC = 24, and leg AB = 5; required the hyp. AC.

OPERATION BY PROB. III. EVOLUTION.

```
        5
        5
485)25.00(.515 + 24 = 24.515, hyp.
  5)24.25(
-----------
4901   7500
   1   4901
-----------
49025  259900
       245125
```

Ex. 3. Suppose the leg BC = 32, and hyp. AC = 124; required the leg AB.

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OPERATION BY PROB. IV. EVOLUTION.

```
          32
          32
          ---
          64
         96   124
     248 )1024 (4.2
       4   976   ----
     ----  ----  119.8 = BC.
     244   4800
      42   4796
     ----  ----
     2398     4
```

Prob. II. Of the six parts of any right-angled triangle [the three sides and the three angles] any three being given, except the three angles; to find the rest.

Theor. 2. All cases in right-angled plane trigonometry may be solved, with sufficient accuracy for most practical purposes, without the aid of either tables or instruments, by what may be called *a natural radius*; and which may be found for *any* given angle, thus: divide the square of the complement of the given angle, expressed in degrees and decimal parts, by the said complement increased by 100 degrees, and to the quotient, carried to two decimal places, add $\frac{1}{3}$ of itself, together with the given angle; and the sum will be the natural radius required.

For any angle not exceeding 30 degrees, you may divide three times the square of the given angle by 1000, to the quotient adding 57·3, and the sum will be the natural radius required.—Now, this natural radius will have the same ratio to the hypothenuse, that the angle (in degrees and parts) for which it was found, has to its opposite leg: and the hypothenuse, added to half the greater leg, will have the same ratio to 86 degrees, that the other leg has to its opposite angle, in degrees and parts.

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Ex. 1. Suppose the angle C = 36° 30′, consequently angle A = 53° 30′,* and hypothenuse AC = 430 equal parts of any denomination; required the legs AB and BC.

OPERATION.

53.5
53.5
2675
1605
2675
136.5)2862.25(20.97
2730 6.99
13225 36 5
12285 64.46 nat. rad. for 36° 30′.
9400
9555

64.46 : 420 : : 36.5 : 237.8 = BC.

237.8
× 237.8
420
840)56548.84(73.8
7 5390 346.2 = AB.
770 2648
73 2091
697 55700
38 55456
6932

Ex. 2. Suppose angle C = 30°, consequently angle C = 60°, and leg AB = 56; required the leg BC, and hyp. AC.

* The three angles of any plane triangle are together = 180°, consequently the two oblique angles will be complements of each other.

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OPERATION.

30
30
900
3
1000)2700
57.3
60. nat. rad. for 30°.

30 : 56 : : 60 : 112 = AC.

56
× 56
224 —— 112
10)3136(14.9
214 214 97.1 = BC.
14 996
200 800
49 19600
1951 17859

Ex. 3. Suppose the leg BC = 52, leg BA = 42; required the hyp. and oblique angles.

OPERATION.

42
× 42
104 —— 52
10)1764(14.9
114 114 66.9 = AC
14 624 26 = ½ BC
128 512 29.9 : 86° : : 42 : 38° 53′. = ✓ C.
49 11200 51 7 = ✓ A.
1329 11961

Ex. 4. Suppose the hyp. AC = 72, and leg BC = 64; required the leg AB and oblique angles.

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OPERATION.

```
        64
      × 64
144 ——— 72
 30 )4096(39
 114 342  33 = AB
  39  676
  75  675  72
           32
          104 : 86° : : 33 : 27° 17′ = √ C
                          62 .43 = √ A.
```

NOTE: All cases in oblique-angled trigonometry [triangles having no right angle] may be solved by the above theorem, having first drawn a perpendicular from one of the angles to the opposite side or its production, and thus forming two right-angled triangles, either distinct from each other, or one included in the other, according as the perpendicular falls within or without the triangle—observing, that the perpendicular be so drawn, that in one of the right-angled triangles thus formed, two parts, besides the right angle, may be given.

In the case when the three sides are given, the perpendicular may be drawn to the greatest side from its opposite angle, and then the segments of the base, or side on which the perpendicular falls, may be found thus: multiply together the sum and difference of the other two sides, and divide the product by the base; then the quotient will be the difference of the segments, which added to the whole base will give double the greater segment, and this segment substracted from the base will leave the less segment.

The application, however, of the foregoing theorem, to the sundry cases in oblique trigonometry; as well as in solving a variety of questions that might be proposed, relative to the calculation of *heights* and *distances*, &c. may, it is presumed, be left to the student without any *explanatory examples*.

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SECT. II. IN MENSURATION OF SUPERFICIES.

The linear dimensions of superficies are usually taken in feet and inches, in yards, or in perches and parts; and the area, or content, computed in square inches, feet, yards, perches, or acres.

In board-measure, it is customary to multiply the dimensions in feet and inches, without reduction; in this case, termed *duodecimal* multiplication. This operation may be performed by the following rule.—Set the multiplier under the multiplicand, and let the *feet* in the former stand under the lowest denomination of the latter; and, in multiplying, carry one for every twelve, from one denomination to another, placing the first product, or remainder, (carrying the twelves,) of each line of products, under the multiplying figure.

When the feet in the multiplier exceeds 12, you may multiply by the aliquot parts of the feet, as in compound multiplication, and then take parts for the inferior denominations.

Ex. 1. Multiply

F.	IN.		F.	IN.
7	9	by	3	6
	3	6		
	4	0	6	
23	3			
Ft. 27	3	6		

Ex. 2. Multiply

F.	IN.		F.	IN.	
30	10	by	20	8.	☞ 20=4×5
	4				
	123	4			
		5			
1	616	8			
$4=\frac{1}{3}$	10	3	4		
$4=\frac{1}{3}$	10	3	4		
Ft. 637	2	8			

Prob. 1. To find the content of a triangle.

(1.) From one side, and the perpendicular distance of the opposite angle.

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Rule.] Multiply the given side by the perpendicular, and half the product will be the area. * Or, one factor multiplied by half the other, will give the area, or superficial content.

Ex. Suppose one side of a triangular field to measure 40.5 perches, and the perpendicular distance of the opposite angle 32.4 perches. Required the content in acres.

40.5
16.2

810
2430
405

160) 656.1 (4.4 Acres.

(2.) From the three sides.

Rule.] Add the three sides together, and take half the sum, from which subtract each side, severally: multiply this half sum and three remainders, continually; and the square root of the product will be the area.

Ex. Suppose the three sides of a triangular field to measure 36, 40, and 50 perches, respectively. Required the content in acres.

OPERATION—BY THE AID OF LOGARITHMS.

36
40
50

2)126 Logs.

63 - - - - - 1.79934
27 - - - - - 1.43136
23 - - - - - 1.36173
13 - - - - - 1.11394

2)5.70637

160)713.15 - - - - - 2.85318

4.45 Acres.

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Prob. 2. To find the content of any four sided figure.

(1.) When the opposite sides are respectively *parallel* to each other. [*A parallelogram.*]

Rule.] Multiply any one of the sides by the perpendicular distance of the opposite side, and the product will be the content.

Ex. Suppose the length of an inch-board = 14 feet 7 inches, and its breadth 15 inches. Required the content in feet and inches, board-measure*.

```
14 7
   1 3
-------
 3 6 9
14 7
-------
Ft. 18 1 9
```

(2.) When two of the sides only are parallel to each other, [*a trapezoid.*]

Rule.] Multiply the sum of the two parallel sides by their perpendicular distance, and the product will be double the area.

Ex. Suppose a two-inch plank, 10 feet 4 inches long, 15 inches broad at one end, and 10 inches at the other. Required the content in feet, &c. board-measure.

```
10 4
   2 1
-------
  10 4
20 8
-------
Ft. 21 6 4
```

(3.) When no two sides are parallel, [*a trapezium.*]

Rule.] Multiply one of the *diagonals* by the sum of the perpendicular distance of the two opposite angles, and the product will be double the area†.

* A foot, board-measure, is considered as equivalent to a foot long, a foot broad, and an inch thick.

† This rule is applicable to *all* four-sided figures.

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Or, you may measure all the sides, and one of the diagonals, dividing the figure into two triangles; and then, in each of these there will be given the three sides, from which the area may be computed, as in *Prob.* 1. *Case* 2.

Ex. Suppose one of the diagonals of a four-sided field to measure 126 perches, and the perpendicular distances of the two opposite angles, 40 and 50, respectively.—Required the area in acres.

126
90

2)11340

160) 5670(35 acres, 70 perches.

NOTE. Any *polygon* may be conceived to be divided into trapeziums and triangles; and then their proper dimensions being respectively taken, and areas found by the above problems, the sum will give the area of the whole figure.—Thus the content of any field, or piece of land, of moderate dimensions, may be found with great accuracy, without the aid of any other instrument, except the common measuring-chain.

Prob. 3. To find the content of any regular polygon.

Rule.] The areas of all similar superficies, [those of the same shape or figure, and differing from each other only in magnitude,] are to one another as the *squares* of their *like* sides; therefore, if the square of the side of any polygon, be multiplied by the area of a similar polygon, whose side is *one*, (found in the following table,) the product will be the content.

TABLE.

No. of sides.	*Names.*	*Mult.'s.*
3	Trigon	0.433013
4	Tetragon	1. - - - -
5	Pentagon	1.720477
6	Hexagon	2.598076
7	Heptagon	3.633912
8	Octagon	4.828427

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Ex. Required the area of a regular hexagon whose sides = each 20 feet.

$20 \times 20 \times 2.598076 = 839.23$ feet, the area.

Prob. 4. Of the diameter, circumference, and area of a circle, any *one* being given, to find the other two.

The areas of circles, are to one another as the *squares* of their like sides or dimensions. The diameter of a circle is to its circumference—

as 7 : 22, nearly.

Or as 113 : 355 } more accurately;
Or as 1 : 3.1416 }

And half the diameter × half the circumference = the area.

Hence, if the diam. be 1, the circum. will be 3.1416, and the area = (3.1416÷4) .7854.

If the circum. be 1, the diam. will = (1 ÷ 3.1416) .3183, and the area = (.3183÷4) .07958.

If the area be 1, the sq. diameter will = (1÷.7854) 1.27324, and the sq. circum. = (1÷.07958) 12.56636.

Hence the following theorems—

1. Diam. × 3.1416 = circum.
2. Circum. × .3183 = diam.
3. sq. Diam. × .7854 = area.
4. sq. Circum. × .07958 = area.
5. Area × 1.27324 = sq. diam.
6. Area × 12.56636 = sq. circum.

Ex. 1. Suppose the diam. = 72 inches. Required the circumference and area.

$72 \times 72 \times .7854 = 4071.5136$, the area.

$3.1416 \times 72 = 226.1954$, the circum.

Ex. 2. Suppose the circum. = 1 mile, (5280 feet.) Required the diameter and area.

$5280 \times .3183 = 1680.624$, the diam. in feet.

$5280 \times 5280 \times .07958 = 2218563.072$, area, in sq. feet.

Ex. 3. Suppose the area = 1 acre, (160 sq. perches.) Required the diam. and circum.

$1.27324 \times 160 = 203.7184$, and $\surd$ = 14.27 perches, the diam.

$12.56636 \times 160 = 2010.6176$, and $\surd$ = 44.85 perches, the circum.

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SECT. III. IN MENSURATION OF SOLIDS.

Definitions.

1. A *parallelopipedon* is a solid contained under six rectangles, the opposite sides being equal, similar, and parallel. If the sides are all equal squares, the solid is called a *cube*.

2. A *prism* is a solid contained under several planes, its bases being equal, similar, and parallel right-lined figures, and its sides, parallelograms. Prisms are particularly denominated from the figure of their bases, as *triangular* prisms, *square* prisms, *hexangular* prisms, &c.; and are said to be *regular* or *irregular*, according as their bases are regular or irregular polygons.

3. A *cylinder* is a round solid, of the same thickness throughout, whose bases are equal and parallel circles.

4. A *pyramid* is a solid contained under a polygonal base, and several plane triangles terminating in a point called the *vertex*.

5. A *cone* is a tapering solid, having a circular base, and terminating in a point called the *vertex* of the cone. A right line drawn from the vertex to the center of the base, is called the *axis* of the cone: When this is at right angles to the base, the solid is called a *right* cone; but when at oblique angles, it is called an *oblique* cone.

6. The *frustum* of a tapering solid, is the lower part, cut off by a plane, parallel to the base.

7. A *prismoid* is the frustum of a wedge-like solid, its bases being parallel, but dissimilar rectangles.

8. A *cylindroid* is the frustum of a cone-like solid, its bases being parallel, but unequal ellipses; or, one a circle and the other an elipsis.

9. A *sphere*, or *globe*, is contained under a curved surface, every part of which is equidistant from a point within the same, called the center.

☞ Many other regular solids might be enumerated, but the above are those which chiefly occur in Practical Mensuration.

Prob. 1. To find the content of any solid, having equal, similar, and parallel bases, and of the same thickness or dimensions throughout.

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Rule.] Find the area of the base, according to its proper figure, and this area multiplied by the height or length, will give the solid content.

Ex. 1. Suppose a solid, of which the bases are equal and parallel right-angled parallelograms, [*a parallelopipedon,*] the length of the base being 14 inches, and the breadth 4 inches; and the length of the solid 10 feet.—Required the content in feet, board-measure.

$14 \times 4 = 56$, area of base in square inches.
$56 \times 10 = 560$, solid content in feet, board-measure.

Ex. 2. Suppose the dimensions of a canal, as follows —width at surface $33\frac{1}{3}$ feet, width at bottom 20 feet, and depth 4 feet: How many cubic yards of earth will be thrown out in digging a mile in length of this canal?

$(33.5+20) \times 2$ [$\frac{1}{2}$ depth] $= 107$, area of section in sq. feet.
$107 \times 1760 \div 9 = 20924\frac{4}{9}$, cubic yards.

Ex. 3. Suppose the length of a cylindrical rolling-stone 5 ft. 3 in., and the circumference 6 ft. 6 in. Required the content in cubic feet.

$6.5 \times 6.5 \times .07958 = 3.362$ area of base.
$3.362 \times 5.25 = 17.65$ cubic feet, the content.

Prob. II. To find the content of a tapering solid [pyramid or cone].

Rule.] Find the area of the base, according to its proper figure, and this area multiplied by $\frac{1}{3}$ of the perpendicular height of the solid, will give its content.

Ex. 1. Suppose each side of the base of a square pyramid = 14 inches, and its perpendicular height = 60 inches. Required its content in cubic inches.

$14 \times 14 = 196$, area of base.
$196 \times 20 = 3920$, cubic inches content.

Ex. 2. Suppose the diameter of the base of a cone = 20 inches, and its perpendicular altitude = 14 feet. Required the content in cubic feet.

$20 \times 20 \times .7854 = 314.16$, area of base in sq. in.
$314.16 \times 1\frac{4}{3} \div 144 = 10.08$ cubic feet.

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Prob. III. To find the content of the frustum of a tapering solid.

Rule.] To the rectangle of the greater and less sides, or diameters, or circumferences, of the bases, add $\frac{1}{3}$ of the square of their difference, and the sum will be the square of a *mean* side, or diameter, or circumference; from which, according to its proper figure, find the area of a mean base; and this multiplied by the height or length of the solid will give the content.

Ex. 1. Suppose each side of the greater base of the frustum of a regular triangular pyramid to measure 12 feet, each side of the less base, 9 feet, and the height, 10 feet. Required the content in cubic feet.

$12\times9+3=111$, square of mean side.
$111\times.433013=48.064$, area of mean base.
$48.064\times10=480.64$ cubic feet, the content.

Ex. 2. Suppose the diameter of the greater base of the frustum of a cone=40 inches, that of the less base =18 inches, and the height=32 inches. Required the content in cubic inches.

$40\times18+161\frac{1}{3}$ ($\frac{1}{3}$ sq. of diff.) $= 881\frac{1}{3}$, sq. of mean diam.
$.7854\times881\frac{1}{3}=629.1992$, area of mean base.
$629.1992\times32=22150$ cubic in., the content.

Prob. IV. To find the content of a prismoid, or of a cylindroid.

Rule.] To the length of the greater base add half the length of the less base, and multiply the sum by the breadth of the greater base, reserving the product. Then to the length of the less base add half the length of the greater base, and multiply the sum by the breadth of the less base; to this product add that before reserved, and the sum will be the triple square of a mean side; from which, according to the proper figure, find the triple area of the mean base, and this multiplied by $\frac{1}{3}$ the height or length of the solid, will give the content.

NOTE. This rule may be applied in finding the content of whole pyramids or cones, or their frustums, or of whole wedge-like solids. In a whole pyramid or cone the length and breadth of the less base = 0; in the

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frustum of a pyramid or of a cone, the length and breadth of the bases, will be respectively equal to each other; and in a wedge-like solid (terminating in an edge) the breadth of the less base will = 0.

Ex. 1. Suppose the length of the greater base of a prismoid = 30 in. and its breadth = 18 in.; the length of the less base = 24 in., and its breadth = 14 in., the length or height of the solid being 48 in. Required the content in cubic inches.

$(30+12)\times 18=756$
$(24+15)\times 14=546$

$1302\times 16\,(\frac{1}{3}\text{ of }48)=20832$, the content.

Ex. 2. Suppose the greater base of a cylindroid to be an ellipsis; its transverse diameter =40 in., and its conjugate = 30 in., the less base a circle whose diameter = 25 in., the perpendicular height being 84 in. Required the content in cubic inches.

$(40+12.5)\times 30=1575$
$(25+20)\times 25=1125$

$2700\times .7854=2120.58$
$2120.58\times 28=29276$ cubic inch. the content.

Ex. 3. Suppose the length of the base of a wedge-like solid = 12 in. and its breadth 3 in., the length of the edge = 8 in., the length of the solid being 15 in. Required the content in cubic inches.

$(12+4)\times 3=48$
$48\times 5=240$ cubic in. content.

Prob. V. To find the content of a sphere or globe.

Rule.] Multiply the square of the diameter, (or axis,) by 3.1416, and the product will be the superficial content; and this multiplied by $\frac{1}{6}$ of the axis will give the solid content. Or, multiply the cube of the diameter by .5236, and the product will be the solid content.

Ex. Required the superficial, and the solid content of a sphere, whose axis = 12 feet.

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$12\times12\times3.1416=452.3904$, the superficial content.
452.3904×2 (or, 5236×1728)$=904.7808$, the solid cont.

Prob. VI. To find the superficial content of any given regular solid.

General Rule.] Consider the species and number of plane figures, either composing the whole surface; or, to which it is equal; then find their areas separately, and their sum will be the superficial content required.

PARTICULARLY.

1. The superficial content of any of the solids in Prob. I., will, besides the two bases, = the rectangle of the length of the solid, into the perimeter of its base.

2. That of any of the solids in Prob. II., will, besides the area of its base, = the rectangle of half the slant height of the solid, into the perimeter of the base.

3. That of any of the solids in Prob. III., will, besides the two bases, = the rectangle of half the slant height, into the sum of the perimeters of its bases.

4. That of a prismoid will, besides the two bases, = the rectangle of the sum of the lengths of the two bases, into their distance + the rectangle of the sum of their breadths, into *their* distance.

5. That of a cylindroid will, besides the two bases, = the sum of the circumferences of the two bases, into half the mean of several equidistant slant heights.

6. That of a sphere=the rectangle of its diameter and greatest circumference.

Ex. 1. Required the whole superficial content of a cone, of which the circumference of the base = 12 feet, and its slant height = 10 feet.

$12\times12\times.07958=11.45952$, area of base.
$12\times5\quad=72.$ area of curve surface.

83.46, superficial content required.

Ex. 2. Required the whole superficial content in sq. feet, of a dough-trough, in form of a prismoid; the length of the bottom, being 3 feet, and breadth 2 feet;

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length of the open top, $3\frac{3}{4}$ feet, and breadth $2\frac{1}{2}$; slant height of the sides 2 feet, and of the ends 2.2 feet.

$3 \times 2 =$ - - - - 6.
$(3+3.75) \times 2 =$ - - 13.50
$(2+2.5) \times 2.2 =$ - 9.9

29.4 sq. ft., superficial cont. req'd.

Prob. 7. To find the content of any irregular solid, of moderate size.

Rule] I. Take an open vessel of sufficient capacity to contain the given body, which fill with water to the point of running over. 2. From this vessel, by means of a cyphon, or otherwise, draw off, into a separate vessel, a little more water than the bulk of the body to be measured, and immerse the body in the remaining water, keeping it down to the bottom, when necessary, with a small stick, or piece of wire, &c. 3. Fill up the vessel again, with the water that had been drawn off, and weigh the remainder of this water, then divide this weight, in ounces, avoirdupois, by 1000, and the quotient will give the weight required in cubic feet and decimal parts*.

Ex. A body of an irregular figure, was found, by the above process, to occupy a space equal to 140 lbs. 12 oz. avoir. of water. Required its solid content.

Lbs. 140,12 = 2252 oz. = 2.252 cubic feet.

SECT. IV. IN GAUGING.

In gauging, the dimensions of the vessel are taken in inches and tenths, and the capacity computed in gallons or bushels.

Gallons are of three kinds, viz. the wine gallon, containing 231 cubic inches; the ale gallon, containing 282; and the dry gallon, containing 268.8; the bushel, (8 dry

* It has been found, by repeated experiments, that a cubic foot of pure water, at a mean temperature (55°), weighs exactly 1000 ounces, avoir., and any moderate change, in temperature, will but very insensibly affect the accuracy of the above result.

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gallons,) consequently, containing 2150.4 cubic inches.

The wine and ale gallons, are to one another, very nearly as 9 to 11.

By the area of any given superficies *in gallons*, is meant the content of that superficies considered as a solid, *one* inch in thickness; and this may be found by dividing the superficies in square inches, by the cubic inches in a gallon.

The area, in gallons, of any regular polygon, or of a circle, whose side, or whose diameter, or circumference, is *one* inch, is called the *multiplier*, for that figure; since being multiplied into the *square* of the side of any similar polygon, or into the square of the diameter, or of the circumference of any circle, the product will be the area, in gallons, of that polygon, or circle.

TABLE OF MULTIPLIERS.

Names of Figures.	*Mult.'s for W. Gallons.*
Trigon. - - - -	.001874
Square. - - - -	.004329
Pentagon. - - -	.007448
Hexagon. - - -	.011247
Heptagon. - - -	.015731
Octagon. - - -	.020902
Circle, (diam.)	.0034
Circle. (circum.)	.000344

Prob. 1. To find the capacity of a vessel in form of any regular solid, occurring in the foregoing problems of mensuration.

General Rule.] Find the *square* of the side of the base, or of the mean base, if a polygon, or of the diameter, or circumference, if a circle; according to the appropriate rule in mensuration of solids; which multiply by the proper multiplier, according to the figure of the base, or mean base, contained in the above table, and then this product, (which will = the area of the base, in

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W. G.) multiplied by the length or depth of the vessel, in inches, will give the capacity in wine gallons.

Ex. 1. Suppose a vessel in form of a cylinder; the diameter at each end = 40 inches, and length = 50 inches. Required the capacity in wine gallons.

$40\times40\times.0034 = 5.44$, area of base in W. G.
$5.44\times50 = 272$, W. G. capacity.

Ex. 2. Suppose a cistern in form of a parallelopipedon; the length = 72 inches, breadth = 60 inches, and depth = 36 inches. Required the capacity in ale gallons.

$72\times60 = 4320$, the sq. of the side of a mean base.
$4320\times.004329\times36 = 673.246$ W. G.
11 : 9 : : 673.246 : 550.8 A. G. the capacity.

Ex. 3. Suppose a vessel in form of the frustum of a cone (a churn, lye-tub, &c.); diameter of the greater base = 36 inches, diameter of lesser base = 28 inches, and depth = 48 inches. Required the capacity in W. G.

$36\times28+21\frac{1}{4}$ ($\frac{1}{3}$ sq. diff.) $= 1031\frac{1}{3}$, sq. of mean diam.
$1031\frac{1}{3}\times.0034\times48 = 168.1$ W. G. capacity.

Ex. 4. Suppose a bathing tub, (bounded by a curve surface,) of the following dimensions, viz. length of bottom = 72 inches, mean of several equidistant breadths = 30 inches, length of the top = 80 inches, mean of several equidistant breadths = 34 inches, and depth = 33 inches. How many gallons, ale-measure, will this vessel contain?

☞ Taking the *mean* breadths of the top and bottom, the vessel may be considered as a *cylindroid*, with elliptical bases.

$(80+36)\times30 = 3480$
$(72+40)\times34 = 3808$
$7288 \times .0034\times11 = 272.57$ W. G.
$= 223.01$ A. G.

Prob. 2. To find the capacity of a *cask*, (hogshead, tierce, barrel, &c.) from *three* dimensions, viz. bung-diameter, head-diameter, and length; all taken inside of the cask, or reduced to inside-measure.

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Rule.] Multiply the difference between the bung and head diameters, by .6, to the product adding the head diameter, and the sum will be a *mean* diameter, or that of a cylinder, of the same length and capacity with the given cask; from which the capacity will be found by the foregoing problem.

The following table will greatly facilitate the calculations in cask gauging*.

* Authors on gauging generally give for multipliers .7, or .67, or .64, or .6, to be used, respectively, according to the greater or less curvature of the staves between the head and bung; but the last (.6) has been found to agree best with actual measurement, in the generality of casks. From these dimentions, however, and where the *variety*, or form, is to be determined merely by the eye, great accuracy, in finding the capacity, is not to be expected.

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TABLE OF THE AREAS OF CIRCLES, IN WINE-GALLONS.

Diam.	Area.	Diff.	Diam.	Area.	Diff.
10	0.340		26	2.298	
		71			181
11	.411		27	.479	
		79			187
12	.490		28	.666	
		85			193
13	.575		29	.859	
		91			201
14	.666		30	3.060	
		99			207
15	.765		31	.267	
		105			215
16	.870		32	.482	
		113			221
17	.983		33	.703	
		119			227
18	1.102		34	.930	
		125			235
19	.227		35	4.165	
		133			241
20	.360		36	.406	
		139			249
21	.499		37	.655	
		147			255
22	.646		38	.910	
		153			261
23	.799		39	5.171	
		159			269
24	.958		40	.440	
		167			275
25	2.125		41	.715	
		173			281
26	.298		42	5.996	

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NOTE. When the diameter is in inches and tenths; then, for the tenths you are to add a proportional part of the difference for one inch: Thus, the area for 15.4 = .765+.042 (.105×.4) = .807.

Ex. 1. Suppose the bung-diameter of a cask = 32 inches, head-diameter = 24 inches, and length 40 inches. Required the capacity in W. G.

32
24

8×.6+24 = 28 8, mean diam. (Tab. num. 2.82.)
2.82×40 = 112.8 W. G. capacity.

Ex. 2. Required the content in W. G. of a pipe of wine; bung-diam. = 27.5 inches, head-diam. = 21.2 inches, and length = 48.7 inches.

27.5
21.2

6.3×.6+21.2=24.98, mean dia. (Tab. num. 2.122)
2.122×48.7 = 103.8 W. G. content.

Prob. 3. To find the capacity of a cask from *four* dimensions, viz. the length, the bung-diameter, the head-diameter, and a middle-diameter, equidistant from the head and bung*.

Rule.] Add into one sum, the areas, in gallons, of the circles corresponding to the bung-diameter, the head-diameter, and to twice the middle-diameter, or, which is the same thing, four times the area corresponding to the middle diameter, then multiply this sum by $\frac{1}{7}$ of the length of the cask in inches, and the product will be the capacity in gallons†.

* The dimensions may be taken with great ease, and sufficient accuracy, outside of the cask, by means of a pair of *sliding-calipers*; deducting for the *thickness* of the cask, at the places of measurement.

† The above rule will give the capacity of *any* cask, whatever may be its form or degree of curvature, generally true to *one-tenth* of a gallon in the hundred, and from no other rule, with only three dimensions, can the capacity be depended on within less than about *one* gallon in the hundred.

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Ex. 1. Suppose the length of a cask 35.3 inches, bung-diameter 32.2 inches, head-diameter 27.7 inches, and middle-diameter 31. inches. Required the capacity in W. G.

Bung 32.2 - - area per Tab. - -	3.526
Head 27.7 - - - - - - - - - - -	2.610
Twice mid. 62.0 - - - - - - - - - -	13.068
	19.204
	35.3
	57612
	96020
	57612
	6)677.9012
	112.98 W. G. cap.

Ex. 2. Suppose the length 26 in., bung diam. 20, head-diam. 17, and middle-diam. 19.2. Required the capacity in W. G.

20 - - - -	1.360
17 - - - -	0.983
38.4 - - -	5.014
	7.357
	26
	44142
	14714
	6)191.282
	31.88 W. G. capacity.

NOTE. The capacity of *any* vessel of moderate size, may, with great accuracy, be found by the following process.—Weigh the vessel, both when full of water and when empty; and then the latter subtracted from the former will leave the weight of the water alone. Divide this weight, in ounces avoirdupois, by 1000, and the quotient will be the capacity of the vessel in cubic feet, and this multiplied by 7.48 (1728 ÷ 231) the gallons wine measure in a cubic foot, will give the capacity in

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W. G. Or, what will amount to the same thing, divide the weight of the water in pounds, by 8.355, and the quotient will give the capacity in W. G.

Ex. Suppose the weight of a vessel when full of water = 1010½ *lbs.*, and when empty = 100 *lbs.* Required its capacity in W. G.

1010.5
100

8.355)910.5(109 W. G. the capacity.

☞ If it were enjoined on coopers, by the proper authority, to mark the *tare* (weight of the empty cask) on the head of the cask, when intended to hold liquor, then, the specific gravity of the liquor being known (and this, as will hereafter be shown, is very readily obtained) the business of gauging would be reduced to the simple operation of weighing, and that with a degree of accuracy unattainable by any other means.

Prob. IV. To find the ullage of a cask; that is, the quantity it wants of being full.

Case 1. When the cask is lying on its side, with its axis parallel to the horizon.

Rule.] Divide the dry inches of the bung-diameter, by the whole bung-diameter, to two places of decimals in the quotient; multiply the quotient by the corresponding multiplier in the following table; and then that product multiplied by the whole capacity of the cask will give the ullage required.

Table.

Quot's	.04	.07	.09	.14	.20	.24	.31	.40	.51
Mult's	.2	.3	.4	.5	.6	.7	.8	.9	1.

Ex. 1. Suppose the bung-diameter = 31.4 in., of which 7.8 dry, and the whole capacity 122.4 gallons. Required the ullage.

31.4)7.8(.25 (multr. .7)
.7

.175×122.4=21.4 gallons, ullage.

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Ex. 2. Suppose the bung-diam. = 28.7 in., of which 10 dry, and the whole capacity 78 gallons. Required the ullage.

28.7)10(.35 (mult'r .8 nearest)
.8

.280×78=22 gallons, ullage.

Case 2. When the cask is standing on one end, with its axis perpendicular to the horizon.

Rule.] Divide the dry inches of the length, by the whole length, to three places of decimals in the quotient; then, when the quotient does not exceed .250, multiply it by .9; but when it exceeds .250, multiply it by 1.1, and from the product subtract .05; and then the result multiplied by the whole capacity, will give the ullage required.

Ex. 1. Suppose the length of the cask = 48 in., of which 8.5 dry, and the whole capacity, 104.7 gallons. Required the ullage.

48)8.5(.177
.9

.1593×104.7=16.7 gallons, ullage.

Ex. 2. Suppose the length = 38 in., of which 15.7 dry, and the whole capacity 120 gallons. Required the ullage.

38)15.7(.413
1.1

.4543
.05

.4043 × 120=48.5 gallons, ullage.

NOTE. If, in either case, the cask be less than half full, then work, as above, with the *wet* inches, and the result will give the quantity of liquor *in* the cask.

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APPLICATION OF ARITHMETIC IN THE SOLUTION OF SELECT PROBLEMS IN NATURAL PHILOSOPHY.

SECTION I.

In finding specific gravities.

Def. The specific gravity of a body is its relative weight compared with that of an equal bulk of *pure water*, which is generally employed, as being the most convenient standard of comparison.

Prob I. To find the specific gravity of a solid body.

Case 1. When the body is small, and insoluble in water, as a piece of coin, a gem, or the like.

Process.] 1. Having provided yourself with a small pair of accurate scales, and a small pair of metalic spring-forceps; suspend these forceps by a horse-hair or thread of silk from one of the scales or ends of the balance, and accurately counterbalance it, when immersed in a tumbler of pure water, by any weights in the opposite scale. 2. Find the weight of the given body in air; which may be considered as its *absolute* weight. 3. Place the body in the spring-forceps, and immerse both together in the water, with the counterbalance of the forceps, and the weight of the body in air, in the opposite scale; and then this end of the balance will, in all cases, preponderate. 4. Restore the equilibrium by weights in the opposite scale, which will give the weight of water equal in bulk to that of the body; which, considering water as the standard of comparison, may be called unity. 5. By the rule of proportion say:—As this relative weight of the body : 1 : : its absolute weight : its specific gravity.

Ex. 1. Suppose the weight of a small piece of lead, in air, = 216 grains, or any other equal weights; and its *relative* weight in water (found as above) = 19. Required its sp. gr.

19 : 1 : : 216 : 11.37, the sp. gr. req'd.

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Ex. 2. Suppose the absolute weight of a piece of cedar wood = 1000 grains, and its relative weight (weight of an equal bulk of water) = 1820. Req'd its sp. gr.

1820 : 1 : : 1000 : .548, the sp. gr. req'd.

Case 2. When the substance is in a granulated form, as grains of sand, gold-dust, or the like.

Process.] 1. Instead of the forceps, take a small open-mouthed glass phial, which, being suspended from one end of the balance, immerse in the water, and counterbalance by weights in the opposite scale. 2. Having weighed the substance in air, put it into the phial, and immerse the whole in the water, keeping the counterbalance of the phial, and the weight of the substance, both in the opposite scale. 3. Add weights to the lighter scale till you produce an equilibrium, and these weights will be equal to the weight of an equal bulk of water with that of the given substance. 4. The sp. gr. will then be found as in the former case.

Ex. Suppose a small quantity of gold-dust to weigh in air 650 grains, and the weight of an equal bulk of water, found as above, 45 grains. Req'd its sp. gr.

45 : 1 : : 650 : $14\frac{4}{9}$ sp. gr. req'd.

Case 3. When the body is small, and soluble in water; as a piece of sugar, salt, or the like.

Process.] 1. Place in your forceps a piece of soft bees-wax, or putty, which immerse in water, and by weights in either scale bring it to an equilibrium. 2. Having weighed the given body in air, envelope it in the wax or putty, and, replacing it again in the forceps, immerse it in the water, and by weights, as before, produce an equilibrium; and then proceed as in the former case.

Ex. Suppose a piece of loaf-sugar to weigh, in air, 482 grains, and the weight of an equal bulk of water (found as above) = 470. Required its sp. gr.

470 : 1 : : 482 : 1.0255, sp. gr. req'd.

Case 4. When the body is large, and insoluble in water, as a block of wood, a stone, or the like.

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Process.] 1. Provide yourself with an empty vessel, as a tub, a barrel, or the like, through the side of which, let a gimblet-hole be bored to receive the barrel of a goose-quill, open at both ends, to serve as a vent or *faucit*. This may be placed a little below the mouth of the vessel. 2. Pour water into this vessel till it rises above the faucit, and let the surplus run out. 3. Having placed a bucket, or any other convenient vessel, under the faucit, immerse the body in the water, keeping it down to the bottom, if higher than water, with a small stick, which, for the sake of greater accuracy, may have been dipped into the water, while the surplus was running out, as deep as when employed in keeping down the body. A quantity of water will now be discharged from the faucit equal in bulk to that of the body immersed. 4. Weigh this water, and then proceed as in other cases.

Ex. 1. A slab of marble weighed, in air, 124¼ pounds avoir., and an equal bulk of water was found, by the above process, to weigh 46 pounds. Req'd its sp. gr.

46)124.5(2.7065, sp. gr.

Ex. 2. A cake of yellow-wax weighed, in air, 164 oz. avoir., and an equal bulk of water, 170 oz. Req'd the sp. gr.

170)164(.9647, sp. gr.

Prob. II. To find the specific gravity of any liquid.

Process.] 1. Take any small solid body, that will sink both in water and in the given liquid, without imbibing either, as the stopper of a decanter. 2. Weigh this, both in air and in water, and thus obtain the weight of an equal bulk of water; next, weigh it in the given liquid, and in like manner obtain the weight of an equal bulk of this liquid. 3. By the rule of proportion say—as the former : unity : : the latter : the sp. gr. req'd.

Or thus :] 1. Take a small glass phial, with a ground-glass stopper, and of this find the weight, in air. 2. Fill this phial with pure water, introducing the stopper when the mouth is under water, to prevent an air bubble, and find the weight of the full phial. 3. Empty out the water, and, having dried the phial, fill it, in the same manner, with the given liquid, and find the weight. 4. Sub-

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tract, in both cases, the weight of the empty phial from that of the full, and the remainders will be the relative weights of equal bulks of water and of the given liquid; from which the specific gravity of the latter will be found as above.

Ex. 1. Suppose the weights of equal bulks of water and of spirits of wine, found by either of the above processes, to be 550 and 453, respectively. Req'd the sp. gr. of the S. W.

550)451(.82 sp. gr.

Ex. 2. Suppose the weights of equal bulks of water and of mercury, found by the second of the above processes, (for they could not be well found by the first,) to be 1500 and 20352, respectively. Req'd the sp. gr. of the mercury.

1500)20352(13.568, sp. gr.

SECT. II. IN PROBLEMS RESPECTING GRAVITY, AND THE DESCENT OF HEAVY BODIES.

Gravity may be considered as a universal property of matter, by the continued influence of which all bodies have a tendency to approach each other. It is from this cause that bodies descend towards the earth.

A body near the surface of the earth, if suffered to fall freely from a state of rest, and without considering the resistance of the medium through which it falls, would descend with an equably-accelerated velocity; the velocity acquired being as the time of descent; consequently the spaces descended through in equal, successive, intervals of time, being as the *odd numbers* 1, 3, 5, 7, 9 &c. and the whole spaces descended through, from the commencement of motion, as the *squares* of the times of descent, or as the squares of the last acquired velocity, or as the squares of the mean velocity, which is half the last acquired, or final velocity. It has been found by experiment, that a body, in such circumstances, would descend 16 feet 1 inch (say 16 feet) in the first second of time, and consequently acquire a velocity of 32 feet per second.

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All problems relating to the descent of heavy bodies, may be solved, by the application of the following theorems; in which

t = the time of descent in seconds,
v = the last acquired velocity, in feet per second,
s = the space desended through from a state rest, in feet.

1. $v = 32\,t = 8\sqrt{s}$

2. $t = \frac{v}{32} = \frac{\sqrt{s}}{4}$

3. $s = 16t^2 = \frac{v^2}{64}$

Ex. 1. In what time would a falling body acquire a velocity of 100 feet per second, and through what space must it descend?

Per theor. 2, $t = \frac{v}{32} = 3\frac{1}{8}$ sec., and

per theor. 3, $s = \frac{v^2}{64} = 156\frac{1}{4}$ feet.

Ex. 2. Through what space would a heavy body descend in 10 seconds, and what velocity would it acquire?

Per theor. 3, $s = 16\,t^2 = 1600$ feet, and
per theor. 1, $v = 32\,t = 320$ feet per second.

Ex. 3. What velocity would a heavy body acquire in descending a mile (5280 ft.) and in what time?

Per theor. 1. $v = 8\sqrt{s} = 581.28$ ft. per second, and

per theor. 2. $t = \frac{\sqrt{s}}{4} = 18.16$ seconds.

Note. 1. A body projected directly upwards, will ascend with a continually decreasing velocity, till it arrives at a state of rest; when it will immediately begin to descend; the times of its ascent and descent being equal: And, even if it be projected *obliquely* upwards, it will obey the same law, and calling *half* the time of flight, t, its greatest height, s, may be found by theor. 3.

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NOTE 2. A body descending by the force of gravity through a resisting medium, as the air, or water, will have its descent more and more retarded by the resistance of the medium, and that in proportion to the square of the velocity, and the density of the medium : so that the velocity, if long enough continued, would at length become uniform, and this will take place so much the sooner according as the surface of the body is great in proportion to its weight; and also according to the resistance of the medium. Hence it may be observed that a flake of snow descending through the air, and a small stone descending through the water, very soon acquire a uniform velocity.

NOTE 3. If the body slide down an *inclined plane*, [one making any angle with the horizon,] without friction or resistance, its velocity at any point of the plane will be equal to what it would have acquired in descending freely through the same vertical height; and the time of sliding down the plane, will be to that of falling freely through its vertical height, as the length of the plane to its height.

NOTE 4. The descent of water in the channel of a river, or canal, or through a close pipe, will be subject to the same laws as that of a heavy body sliding down an inclined plane.

The velocity of a current of water in an open channel is greatest at the surface in the middle of the channel; least at the bottom and margin or sides (from the effect of friction); and the *mean* velocity, or that from which the expense of water is to be computed, is always *nearer* the bottom than the surface. The velocity at the surface is easily ascertained by observation, and from this the velocity at the bottom, and the mean velocity, may be computed by the following rule.

From the square-root of the greatest superficial velocity (which may be expressed in inches per second) take unity, and the square of the remainder will be the velocity at bottom; then half the sum of these two will be the mean velocity. Thus, if the superficial velocity be

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9 inches per second, the velocity at bottom will be 4 and the mean velocity, $6\frac{1}{2}$.

A certain velocity of the water is necessary for the stability of the bed or channel. If the velocity be too great, it will wear it away or tear it up; and if too small, it will permit a sediment from the water to collect at the bottom. It is found by observation, that a velocity of 3 inches per second, at bottom, which corresponds to 7.46 at the surface, will just begin to work upon fine clay, and is perhaps as great a velocity as any canal ought to have*.

SECT. III. IN PROBLEMS RESPECTING THE OSCILLATION OF PENDULUMS.

A pendulum is a body oscillating backwards and forwards, from a center of suspension, called also the center of motion. If the bob or lower extremity of the pendulum move through small, though unequal arches, the times of oscillation will be all nearly equal to each other, and proportional to the square-roots of the length of the pendulum, or distance from the center of suspension to the center of oscillation; or, which is the same thing, the lengths will be as the squares of the times of oscillation. This center, in the bob pendulum, is a little above the center of gravity of the bob; but in a right line it is at two-thirds, and in a small cylinder or prism, at *very nearly* two-thirds of its whole length from the center of motion. A second-pendulum of this kind, as recommended by Mr. Jefferson, when Secretary of State, would, it is believed, be the best *universal standard of measure*, that nature affords. As it would be difficult, if not impracticable, to make the center of suspension of a rod-pendulum exactly coincide with its upper extremity, it may be advisable, in this case, to make the center of suspension at the distance of one-third of the length from the upper extremity, and then the times of oscillation would be exactly isochronous with those of a simple pendulum of two-thirds the length of the rod-pendulum.

* See Dobson's Encyclopædia, Article Rivers.

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The length of the pendulum will be somewhat different in different latitudes: for, both on account of the spheroidal figure of the earth, and its diurnal rotation, the gravity, and consequently the length of the pendulum, will be least at the equator, and gradually increase to the poles. In latitude 40°, the length of the simple second-pendulum, according to the computations of Mr. Emerson, founded on real observation, will be 39.097 (say 39.1) inches, and consequently the rod-second-pendulum = 52.13.

Pendulums are chiefly employed in regulating the going of clocks. These, however, are subject to irregularities; 1st, from changes of temperature; for heat will expand the pendulum, and cause the clock to go slower, and cold will produce the contrary effect. 2d. From imperfection in the wheel-work: for, when the friction in one part of the revolution is less than in another part, the maintaining power, meeting with a less degree of resistance, will cause the arch of vibration of the pendulum to be greater (and thus the clock to go slower) at one time than at another. Irregularities arising from a change of temperature may be corrected by employing two metals which are affected in different degrees by change of temperature; as iron with brass, or zinc, or mercury; and so combined as by their different degrees of expansion and contraction to cause the length of the pendulum always to remain the same. The irregularity arising from the imperfection in the wheel-work, is best corrected by making the bob of the pendulum heavy, and the arch of vibration small: for the vibrations in small, yet unequal arches, though not *perfectly*, are yet very nearly isochronous.

Most questions relating to the oscillations of pendulums, may be solved by the application of the following theorems, in which—

Let t = the time of one oscillation in seconds,
n = the number of oscillations in a minute,
l = the length of the pendulum from the center of suspension to the center of oscillation, in inches.

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1. $l = 39.1 \times t^2 = \frac{(39.1 \times 60 \times 60)}{n^2} = \frac{140760}{n^2}$

2. $t = \sqrt{\frac{l}{39.1}} = \frac{60}{n}$

3. $n = \frac{60}{t} = \sqrt{\frac{140760}{l}}$

Ex. 1. Required the length of a pendulum, that shall vibrate half seconds.

Per theor. 1. $l = 39.1 \times .25 = 9.775$ inches. Ans.

Ex. 2. In what time will a pendulum of 50 feet (600 inches) in length make a single vibration, and how many would it make in a minute?

Per theor. 2. $t = \sqrt{\frac{600}{391}} = 3.9$ sec., and

per theor. 3, $n = \sqrt{\frac{140760}{600}} = 15.3$. Ans.

Ex. 3. What must be the length of a pendulum, to make 13 vibrations in a minute?

Per theor. 1, $l = \frac{140760}{13 \times 13} = 832.9$ inches. Ans.

Ex. 4. On examining the pulse of a person in a fever, I observed that it kept time with the oscillations of a pendulum of 20 inches in length. How many pulsations in a minute?

Per theor. 3, $n = \sqrt{\frac{140760}{20}} = 83.8$. Ans.

Ex. 5. What length must a pendulum be to make 12 oscillations, in the same time that one of 9 inches in length makes 30? The lengths of pendulums are to one another as the squares of their times of oscillations *directly*; or as the squares of their oscillations, in the same time, *inversely*.

$30^2 : 9 : : 12^2 : 5\frac{1}{4}$ inches, inversely—Ans.

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SECT. IV. IN FINDING THE HEIGHTS OF MOUNTAINS, &c., BY THE BAROMETER, WITH THE AID OF THE THERMOMETER.

The barometer, in its most simple, and perhaps best form, is a long glass tube, close at one end and open at the other. This tube being filled with mercury, from which the air is to be expelled by boiling it in the tube, is closed at the open end by the finger, then inverted, and set vertically in a basin of the same fluid. The finger being now withdrawn, the mercury will subside; and, leaving a vacuum in the upper part of the tube, a column of it will be supported by the pressure of the atmosphere acting on the lower end of the tube, equal in weight to an entire column of the atmosphere, of the same base with that of the mercurial column. This column, at the surface of the earth, not far above the level of the sea, is found to vary from 28 to 31 inches; and at a mean would be equal to a pressure of about $14\frac{1}{2}$ pounds, avoir., on every superficial inch.

A thermometer is an instrument for indicating the temperature of the air, or any other medium to which it is applied.

The most common kind of thermometer, is a glass tube, of small bore, with a hollow ball or bulb at one end. This bulb, with part of the tube, is filled with mercury or spirit of wine; and the other end of the tube hermetically sealed. A change of temperature will then be indicated by the rising or falling of the fluid in the tube, occasioned by its expansion or contraction by heat or cold.

There are two fixed or permanent points of temperature, from which all the others are found. One is that at which water begins to freeze, or ice to thaw, termed the freezing point; and the other, that at which water boils, under a mean pressure of the atmosphere. The first is marked 0°, and the last 80°, on Reaumur's scale; but on Fahrenheit's scale, the first is marked 32°, and the last 212°. On the centigrade scale, the first is marked 0°, and the last 100°.

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The above instruments may be very advantageously applied in ascertaining the height of mountains, or other elevations, as follows:—

Let two persons, each furnished with a barometer and a thermometer, observe, both at the top and bottom of the mountain, &c. the height of the mercury, (in inches and decimal parts,) in the barometer; also, by the thermometer, the temperature of the mercury in the barometer, and that of the neighbouring air, in the shade. Then, from these observations, the height may be calculated by the following rule:—

1. Let the height of the mercury in the *warmer* barometer be corrected on account of their difference of temperature, by the following proportion, viz.

As 9600: the diff. temp. in degrees of Fahr. scale:: the height of the mer. in the *warmer* bar.: the correction to be subtracted therefrom.

2. Take the common logarithms of the two heights of mercury, thus corrected, and subtract the less from the greater; then the first four figures, or places, on the left hand, will be the height of the mountain, &c. in fathoms, (each six English feet,) and the remaining figures on the right, decimal parts.

3. This height is, however, to be corrected on account of the difference of temperature in the neighbouring air, of the two barometers, as follows—Take the difference between the mean of these temperatures and 31°, and then by the rule of proportion, say—as 435: this diff.:: the height before found: the correction; which is to be added or subtracted according as this mean temperature is greater or less than 31°.

Ex. 1. Let the state of the barometers and thermometers, at the top and bottom of an elevation, be as follows, to find its height.

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Ther. detached.	attached.	Barom.
42°	43	25.28—Log. 1.402777
57	57	29.68
9600 :	14::2968:	.04
Mean 49.5		
31		2964—Log. 1.471878
435 ; 18.5 ::		691.01:29.38
		+ 29.38
		720.39 fath.'s, height required.

Ex. 2. Suppose the state of the barometers and thermometers at the top and bottom of a hill, as follows—

Ther. detached.	attached.	Barom.
31	38	26.82—Log. 1.428459
35	41	29.45
9600 :	3::29.45:	.01
Mean 33		
31		29.44—Log. 1.468938
435 : 2 : :		404.79 : 1.86
		+ 1.86
		406.65 fa. Ans.

SECT. V. IN THE SOLUTION OF PROBLEMS RESPECTING THE MECHANIC POWERS.

A *mechanic power* is any machine or contrivance, by the aid of which a greater weight may be raised by a less. Or, more generally, by which the velocity or direction of the weight may be changed.

The *simple* mechanic powers are reckoned *six* in number, viz. the lever, the axis-and-wheel, the pulley-and-tackle, the inclined plane, the wedge, and the screw; for, to one or other of these, single or combined, may most machines, for raising weights, or overcoming resistance, or producing, or changing motion, be reduced.

The *lever*, in a mathematical sense, may be considered as a movable inflexible line, acted upon by two forces, usually denominated the *power* and the *weight*, applied

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to different parts thereof; with the re-action of a third point, called the *fulcrum*, or center of motion.

Levers are distinguished into three kinds, according to the disposition of the power, weight, and fulcrum, viz.

1st. Where the fulcrum is between the power and the weight.

2d. Where the weight is between the fulcrum and the power.

3d. Where the power is between the fulcrum and the weight.

In a *straight* lever, the above three points are considered as in the same right line: but in a *bent* lever, the lines connecting these points make an angle with each other.

In the *axis-and-wheel*, the power acts through the medium of a rope passing round the circumference of the wheel, while the weight to be raised is suspended to a rope passing round the axis, or a wheel of different diameter, and is, in principle, no other than a *perpetual lever*.

In the *pulley-and-tackle*; however differently the parts may be arranged, the effect may be very readily computed by merely considering the number of ropes supporting the weight, and which must be all shortened in raising it.

The *inclined plane* is one making an angle of elevation above the horizon; as the tail-end of a dray, &c., generally employed for the purpose of more easily raising heavy articles.

The *wedge*, is no other than a single or double inclined plane, generally employed in separating the parts of any tenacious substance, as in splitting wood; or in producing great pressure, as in the oil-mill. To this the power is usually applied by the stroke of a mallet, hammer, or the like.

The *screw* is a modification of the inclined plane, and is generally used in producing great pressure; as in the vice, coining-press, &c. In this the power is most frequently applied by means of the lever.

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All questions relating to the mechanic powers, may be easily solved by the application of the following rule.

Consider, from the construction and arrangement of the parts of the machine, what would be the proportion between the velocity of the power and that of the weight, if motion were commenced; for then, the power will be to the weight as the velocity of the weight is to that of the power, when in equilibrio.

This proportion between the power and the weight may be expressed in different terms, according to the mechanic power employed. Thus, in the lever, it will be—

As the distance of the fulcrum from the weight,
Is to its distance from the power;
So is the power,
To the weight.

In the axis-and-wheel.

As the diameter, (or circumference,) of the axis,
Is to the diameter, (or circum.) of the wheel;
So is the power,
To the weight.

In the pulley-and-tackle.

As unity,
Is to the number of ropes supporting the weight;
So is the power,
To the weight.

In the inclined plane.

As the perpendicular height of the plane above the horizon,
Is to the length of its sloping side;
So is the power,
To the weight.

In the wedge, or double inclined plane.

As the thickness of the base or butt of the wedge,
Is to the sum of the lengths of its sloping sides;
So is the power communicated,
To weight to be raised, or resistance to be overcome.

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In the screw.

As the distance between two threads of the same spiral, whether the screw have one or more threads,

Is to the circumference of the circle described by the power;

So is the power,

To the weight to be raised, or pressure to be communicated.

NOTE 1. By alternation or reversion, in the above proportions, any *three* terms being given, the fourth may be found.

NOTE 2. In the above proportions, no allowance is made for *friction* among the parts of the machine: this, however, is in most cases very considerable; sometimes amounting to *one-third* of the effect which would otherwise be produced.

NOTE 3. In the above proportions, the case of *equilibrium* only, between the power and the weight, is considered; and therefore, in order to produce actual motion in the weight, &c., there must be an increase of power, not only sufficient to overcome the friction, but to produce the required velocity in the weight.

NOTE 4. The power of the axis-and-wheel may be increased at pleasure, by making one part of the axis thicker than the other, with a rope winding round the thicker part, while it is unwound from the smaller part, and the weight to be raised attached to a pulley, supported on the bend of the rope; for then, the power will be to the weight, as the *difference* between the diameters of the thicker and smaller parts of the axis, is to the diameter of the wheel, to which the power is applied.

NOTE 5. The power of the screw, also, may be increased at pleasure, by having the threads cut coarser on one part of the screw than on the other, each working in a separate box, one being fixed and the other movable. For, in this construction, the power will be increased in proportion as the *difference* between the threads of the coarser and finer screws is diminished.

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EXAMPLES.

1. Suppose, in a lever, the distance of the fulcrum from the weight = 4 inches, and its distance from the power = 36 inches. What power would sustain, in equilibrio, a weight of 112 pounds?

$$36 : 4 :: 112 : 12\tfrac{4}{9} \text{ lbs. Ans.}$$

2. Suppose, in a lever, two powers employed to support a weight, viz. one power of 3 pounds, at the distance of 20 inches from the fulcrum; and the other, of half a pound, at the distance of 9 inches. What weight, at the distance of 6 inches from the fulcrum, would be a counterbalance to the above powers?

$$\left.\begin{array}{l} 6 : 20 :: 3 : 10 \\ 6 : 9 :: .5 : 0.75 \end{array}\right\} = 10\tfrac{3}{4} \text{ lbs. Ans.}$$

3. In order to find the weight of a heavy beam of timber, lying on the ground, I introduced under one end thereof, a small cylindrical roller of hard wood, shoving the timber an-end on the roller, till the two ends appeared to counterbalance each other. Then making a mark on the timber directly above the vertical diameter of the roller, I again shoved the timber an-end, till the roller occupied a place two feet nearer to the smaller end of the timber than the aforesaid mark. In this position, I applied weights to the extremity of the lighter end, till the equilibrium was again restored. The weights, thus applied, I found to be 118 lbs., and their distance from the roller, (fulcrum,) $26\frac{1}{2}$ feet. Required the weight of the timber.

$$2 : 26.5 : 118 : 1563\tfrac{1}{2} \text{ lbs. Ans.}$$

4. What must be the height of an inclined plane, of 60 inches in length, so that a weight of 224 pounds may be supported on the plane, by a counterweight, (power,) of 30 pounds, acting freely by its gravity?

$$224 : 30 :: 60 : 8\tfrac{1}{28} \text{ inches. Ans.}$$

5. Suppose four horses, (or any other given power,) sufficient to haul a load of 30 hundred weight up an ascent, rising one foot in 28.65 (2°). What weight would

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the same power be able to haul up an ascent of one foot in 11.49 (5°)?

28.65 : 3360 :: 11.47 : 1345 = 12 cwt., 1 lb. Ans.

SECT. VI. IN ASTRONOMICAL CHRONOLOGY.

Definitions, &c.

Chronology treats of *time*, with respect to its measures; including its various distinctions, divisions, and subdivisions.

Time is measured or regulated by the motion of the heavenly bodies; and is accordingly distinguished into solar, (mean, and apparent,) lunar, and sidereal.

Apparent solar noon, at any place, is the moment of time when the sun's center is on the meridian of that place. An apparent solar day is the interval of time between one apparent solar noon and the next. These days are not all equal to each other throughout the year. This inequality arises from two causes—one is, that the earth, moving in an elliptical orbit round the sun, does not describe equal arches in equal times, moving slower or quicker, according as its distance from the sun is greater or less; and, therefore, as the earth, in a solar day, must make one complete rotation round its axis, and so much of another as will correspond with its diurnal arch in the ecliptic; and since these arches are unequal, it follows, that, on this account, the days must be unequal also. The other cause of inequality is, the inclination of the ecliptic to the equator; whence, equal arches in the ecliptic, in which the earth moves, will not correspond with equal arches of the equator, on which time is measured.

Mean solar time is that pointed out by a true timepiece, going with an equable motion throughout the whole year, and the difference between this and *apparent solar time*, or the distance of the sun from the meridian, (15 degrees to an hour,) is called the *equation of time*.—Four times in the year, viz. on the 16th of April, 16th of June, 1st of September, and 25th of December, this

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equation will be nothing; but when greatest it will amount to upwards of 16 minutes.

A *lunar* day is the interval of time between the moon's passing the meridian on any day, and on the next succeeding day. These days are unequal; but, on an average, exceed a mean solar day about 48 minutes.

A *sidereal* day is the interval of time from the passage of any fixed star over the meridian, till it passes the meridian again. These days are all equal, and 3′ 55.9″ of time less than a mean solar day.

A *solar* or *tropical* year is the time in which the earth performs a complete revolution through all the artificial signs of the ecliptic, = 365 days, 5 hours, 48 minutes, 48 seconds. And a *sidereal* year is the time in which the earth performs a complete revolution through the whole circle of stars in the ecliptic, = 365d. 6h. 9m. 17s., being 20m. 29s. greater than the solar year. This difference is occasioned by the *precession* of the equinoxial points, arising from the oblate spheroidal figure of the earth, and the inclination of the ecliptic to the equator.

The *civil* day is, in most nations, considered as beginning at mid-night; the *astronomical* day, at the succeeding noon; and the *nautical* or sea-day, at the preceding noon.

The mean civil year, according to the Julian calendar, contained 365d. 6h. that is, three years of each 365, and the fourth of 366, the supplementary day being added to the month of February, by counting *twice* the 23d day, which in the old Roman calendar was called the *sixth* of the calends of March; and hence this year was called *bis-sextile*, and on another account *leap-year*. This Julian year exceeds the true or solar year, according to which the seasons take place, 11m. 12s. which in 131 years amounts to a whole day; and this constitutes the difference between the Julian, or *old-style*, and the Gregorian, or *new style*.

In the year 325 when the council of Nice appointed the time for the celebration of Easter, (viz. the first Sunday after the full-moon immediately succeeding the time of the vernal equinox,) the equinoxes happened on the

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21st of March and 21st of September. In the year 1582, Pope Gregory XIII. observing that the Julian had got a-head of the solar year, 10 days; ordered that so many should be then struck out of the calendar; and in the year 1753, when the British government adopted the new-stile, 11 days were necessarily struck out. To prevent any irregularity from taking place in future, it was ordered that *three* days should be struck out of the Julian calendar every 400 years, by reckoning 1700, 1800, 1900, 2100, &c. or every centurial year not divisible by 4, a *common* year instead of a *leap-year*, which it would otherwise be.

The first year of the Christian æra was the first after leap-year; hence, if you divide the year of the Christian æra by 4, then, if nothing remains, it will be leap-year, except the centurial years not divisible by 4, which, according to the Gregorian calendar, must be considered as common years. But if any thing remains it will point out the year after leap-year.

For example, the year 1819?

4)1819

454..3d. after leap-year.

In calendars, it was formerly customary to prefix the first seven letters of the alphabet to the several days of the year, successively; that opposite the first day of the year being A, that opposite the second, B, &c. on to G. The same letter, therefore, would continually correspond to the same day of the week throughout the year, and thus the letter corresponding to the Lord's day (Dominicus dies) was called the Dominical letter. A common year containing 52 weeks and 1 day, and a leap-year, 52 weeks and 2 days, the Dominical letter will therefore fall back, in the order of the alphabet, one letter every common year, and two letters every leap-year—and hence the denomination, *leap-year*.

The week-day on the first of January, and thence the Dominical letter, for any given year, may be found thus:

To the given year add its 4th part, rejecting fractions,

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and divide the sum by 7; to the remainder add 3 for the century beginning with 1600; 2 for that beginning with 1700; 1 for that beginning with 1800; and 0 for that beginning with 1900: then the sum will be the number of the day of the week required: and this number taken from 9, will leave the number of the Dominical letter, in the order of the alphabet, viz. 1=A, 2=B, &c.

Ex. 1. On what day of the week did the first of January fall in the year 1776; and what was the Dominical letter that year?

1776
444

7)2220

317 - rem. 1+2=3, Tuesday; 9—3=6, F. Dom. let*.

Ex. 2. Required the day of the week on which the first of January will fall in the year 1877, and the Dom. let. that year.

1877
469

7)2346

335 - - rem. 1+1=2, Monday.
9 — 2 = 7, G Dom. let.

Having the Dominical letter for any year, you may readily find on what day of the week, any given day of any month will fall, by the following distich:

"At Dover Dwells George Brown, Esquire,
Good Christopher Finch And David Friar."

For the initial letter of each of these twelve words, is the letter which, in the calendar, belongs to the first day

* In leap-years, the letter found by the above rule will be the Dom. let. after the month of February, and the succeeding letter of the alphabet will be the Dom. let. for the preceding two months.

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of its respective month, from January (At) till December (Friar). Hence counting from that letter to the Dominical letter, you will have the day of that month on which the first Sunday falls. Thus:—Supposing the Dominical letter for any given year, C; on what day of the week will be the 4th of July?

July, "Good." G, A, B, C=Sunday, the 4th July.

Or, knowing the day of the week on which the year began, you may find the day of the week on which any month will begin, by the following verse.

To range the months in such a way
As gives a diary in a word,—
October rank with new-year's day,
Second May, and August *third*;
Fourth February, March, November;
The *fifth* place let June supply;
Sixth September and December,
Seventh April and July.

NOTE. In leap-years, you must add *one* to all the numbers in the above verse, after the month of February; as in the 2d example.

Ex. 1. Suppose the year to begin on Friday—6th day; being a common year, on what day of the week will May begin?

"*Second* May." *i. e.* the second from Friday, inclusive, or Saturday, will be the 1st of May.

Ex. 2. Suppose as above; but a leap-year.

Third, from Friday; or Sunday, the 1st of May.

A *Cycle*, in chronology, is a certain period of time wherein the same circumstances to which the cycle has a reference, will regularly return. The most noted cycles are the two following, viz. the Solar, and the Lunar or Metonic.

The solar cycle is a period of 28 years, after which the same day of any month will occur on the same day of the week, as on the same year of a former cycle.

The first year of the Christian æra, was the 9th of the solar cycle; hence this rule—Add 9 to the given year,

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divide the sum by 28, and the remainder will be the year of the cycle required—0 corresponding to the 28th or last year of the cycle.

For example, the year 1819?

28)1819+9(65—rem. 8 = year of solar cycle.

The lunar cycle is a period of 19 years, in which the moon's age will be the same, as on the same day of the same month, in the same year of a former cycle.

The first year of the Christian æra was the first of this cycle; hence this rule—Add 1 to the given year, divide the sum by 19, and the remainder will be the year of the cycle required—0 corresponding to the 19th, or last year of the cycle. The year of this cycle is frequently called the *prime*, or *golden number*.

Required the golden number for the year 1819.

19)1819+1(95—rem. 15 = golden number.

The *epact* is the moon's age, reckoned from the change, at the beginning of the year. A lunar year, or 12 lunar months, contains about 11 days less than a common solar year. Hence, should the solar and lunar year begin together, consequently the epact = 0, the next year the epact would be 11, the next 22, the next 3, (33—30,) the next 14, and so on, through the cycle of 19 years; after which the epacts would again return in the same successive order: Hence, this rule—Divide the year of the Christian æra by 19, multiply the remainder by 11, and divide the product by 30, (the nearest whole number of days in a lunar month,) and the remainder will be the epact.

Required the epact for the year 1819.

19)1819(95—rem. 14

11

30)154(5—rem. 4, epact.

The moon's age, on any given day, may be found thus—Add together the epact, the day of the month, and the number corresponding to the month in the following table; and then, the sum, or its excess above 30, will be the moon's age, nearly.

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Table.

Jan.	Feb.	Mar.	Ap.	May,	June,	July,	Aug.	Sep.	Oct.	Nov.	Dec.
Common year.											
0	2	0	2	2	4	4	6	7	8	10	10
Leap year.											
0	2	1	3	3	4	5	7	8	9	10	11

Required the moon's age, on the 12th of Oct. 1819.

4 epact.
12 day of month.
8 number of month.
24 moon's age.

The time of the moon's southing, or passage over the meridian, may be found thus—From her age subtract one-fifth part, and the remainder will be the time of her passage, nearly.

Required the time of moon's passage over the meridian, when 24 days old.

5)24
4.48

19.12 i. e. 7 h. 12 m. A. M.

FURTHER EXAMPLES, IN CHRONOLOGICAL CALCULATIONS.

Let it be required to find: 1. The week-day on the 1st of January: 2. The Dominical letter: 3. The year of the solar cycle: 4. The year of the lunar cycle, (prime or golden number): 5. The epact; for the year 1845: Also, 6. On what day of the week the 4th of July will fall in that year: 7. What will then be the moon's age: And, 8. at what time she will pass the meridian.

4)1845
461
7)2306
(1, 2.) 329...rem. 3
+ 1
4, Wed. 1st of Jan. 9—4 = 5, i. e. *E*, Dom. Let.

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1845
9

(3.) 28)1854(66...rem. 6, year of solar cycle.

(4.) 1845
1

19)1846(97...rem. 3, year of lunar cycle, or golden number.

(5.) 19)1845(97...rem. 2
11
22, Epact.

(6.) July, "*Good.*"...*G*, *A*, *B*, *C*, *D*, *E* = 6, first Sunday, cons. Friday, 4th July.

Or, "*Seventh*...July," 7th from Wed. inclusive, is Tu. 1st July; cons. Frid. 4th.

22 Epact.
4 day of month.
4 number of month.

30—30 = 0, moon's age, or moon changes.
$0-\frac{0}{1} = 0$, or noon, time of passing meridian.

MISCELLANEOUS ARTICLES, FOR THE EXERCISE OR AMUSEMENT OF THE STUDENT IN ARITHMETIC.

I. SELECT PROPERTIES OF NUMBERS.

1. THE sum, or the difference of two *even*, or of two *odd* numbers, will be an *even* number. But the sum, or the difference, of an even and an odd number, will be an odd number.

2. The sum of any number of even numbers, or of an even number of odd numbers, will be even. But the sum of an odd number of odd numbers, will be odd.

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3. If one, or both, of two factors be even numbers, the product will be an even number: but if both be odd, the product will be odd.

4. If any two numbers be, severally, divisible by a third, (without a remainder), their sum, and their difference, will also be divisible by the same.

5. If several different numbers be each divisible by any other number; then their sum will be divisible by the same number, and their product, by the same number, or any power thereof, whose index does not exceed the number of factors.

6. If several different numbers be each divisible by 3 or by 9; then their sum, or the sum of their digits, will also be divisible by the same numbers, respectively.

7. If any number be multiplied by a number divisible by 9, or by 3, then the product, or the sum of its digits, will also be divisible by the same numbers, respectively.

8. If any number be divisible by 3, by 9, or by 11; then its reverse (the same number written backwards) will also be divisible by the same numbers, respectively.

9. The sum of any number consisting of an *even* number of places, and its reverse, will be divisible by 11, and their difference by 9. But if the number of places be *odd*, then the difference of the number and its reverse will be divisible both by 11 and by 9.

10. If any even number be divisible by 9, it will be divisible also by 18, or by any aliquot parts of 18, (6,3,2).

11. Any number, or the sum of its digits, divided by 9 or by 3, will leave the same remainders.

12. If the sum of the digits in the odd places of any number, begining at the place of units, be equal to the sum of the digits in the even places; then, both the whole number and its reverse, will be divisible by 11; their sum will also be divisible by 11, and their difference both by 11, and by 9. When, in the above case, the number and its reverse (divisible by 11,) consist of an *odd* number of places, then the sum of the quotients will be divisible by 11, and their difference by 9. But when the number, as above, consists of an *even* number of places, then the

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difference of the quotients will be divisible both by 11, and by 9; or 99.

SECT. II. SELECT PROBLEMS IN ARITHMETIC.

Prob. I. To find the *sum* of the products of any number of given factors, by simple addition; and that without finding any one of the particular products*.

RULE.

1. Rule out nine columns or vertical spaces, which number at the head 1, 2, 3, &c.

2. In these columns, write down, in succession, the several multipliers, each under the digits of its multiplicand at the head of the column; observing to set the units figure of each multiplier, in such place of the column, as the figure at the top occupies in its multiplicand.

3. Add up the numbers in column 9, and set down the sum; then to this sum add the numbers in column 8, and set down the sum under the former: to this last sum add the numbers in column 7; and so on, with all the other columns.

4. Add together the nine sums before found, and their sum will be that of the products required.

Ex. Required the sum of the products of the following factors, viz.

342× 756
301× 127
89×1634
120× 97
551× 274
6.8× 16.9

* See Transactions of the American Philosophical Society, Vol. I. New Series, No. XII.

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(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
30100	3010	890	89	3420	342	34200		1200
89000	5310		55.1		8900	301		.68
68					6.8	120		
						551		1200.68
								1200.68
								36372.68
								45621.48
								49041.48
								49185.58
								50075.58
								58595.58
								177763.58

Sum of products = 469057.32

Prob. II. To find, when *possible*, the least number, and thence a series of numbers, which being divided by any other given numbers, shall severally leave given remainders.

ILLUSTRATED BY EXAMPLES.

Ex. 1. Suppose the given divisors, 5, 6, 8, 9, 11, and the given remainders, - - - - - - 3, 4, 6, 1, 0, respectively.

OPERATION.

6) 3 - - rem. 3, 2, 1, 0, 5, 4.
25

8)28 - - rem. 4, 2, 0, 6.
90

9)118 - - rem. 1.
11)118 - - rem. 8, 5, 2, 10, 7, 4, 1, 9, 6, 3, 0.
3600

3718, the least req'd dividend.
3960

7678, second greater.

11638, third greater, &c.

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EXPLANATION.

(1.) The first remainder, 3, divided by the first divisor, 5, will evidently leave the required remainder, 3; and if this be divided by the second divisor, 6, it will leave the same remainder, 3. (2.) If to this number you successively add the first divisor, 5, and subtract the second, 6; or, which is the same thing, subtract their difference, 1, adding in the second divisor, 6, when necessary, you will obtain the series of remainders, 3, 2, 1, 0, 5, 4; the last term of which being the remainder required, you are to multiply the number of terms *less one*, 5, by the first divisor, 5, and then adding the product, 25, to the first remainder, you will have 28; a dividend which will answer the conditions of the two first divisors, leaving the respective given remainders, 3, 4. (3.) If the dividend last found, 28, be divided by the third divisor, 8, the remainder will be 4; and then continually adding thereto the least common multiple of the two first divisors, 30, and subtracting the third divisor 8; or, which is the same thing, adding the remainder when 30 is divided by 8, viz. 6; or, subtracting its complement to 8, viz. 2, you will obtain a series of remainders, 4, 2, 0, 6; the last term of which being the required remainder, you multiply the aforesaid common multiple of the two first divisors, 30, by the number of terms less one, in this series of remainders, viz. 3, and the product, 90, added to 28, the dividend before found, will give 118, a dividend which will answer the conditions of the first three divisors. (4.) 118 divided by the fourth divisor, 9, will leave the remainder 1, which being the required remainder, it follows that 118 will answer the conditions of the four first divisors. (5.) Lastly, 118 divided by 11, the last divisor, will leave 8 as a remainder, to which adding continually 8, (the remainder when 360, the least common multiple of the four first divisors, is divided by 11,) or subtracting its complement to 11, viz. 3, you will obtain the series of remainders, 8, 5, 2, 10, 7, 4, 1, 9, 6, 3, 0; then 10, the number of terms less one, multiplied by 360, and the product added to 118 will give 3718; the least number which will answer the conditions of the ques-

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tion. And if 3960, the least common multiple of all the divisors, be continually added to the above number, you will obtain a series of numbers, viz. 3718, 7678, 11638, 15598, &c. each of which will answer all the conditions of the question.

In many examples of this problem, the conditions proposed may not all be possible. This will always be the case when, in the series of remainders, the same number occurs twice, without producing the remainder required, as in the following example;

Ex. 2. Suppose the given divisors, 9, 10, 11, 15.
And the given remainders, 5, 7, 9, 0, respectively.

10) 5 - - rem. 5, 4, 3, 2, 1, 0, 9, 8, 7.
8×9 = 72

11)77 - - rem. 0, 2, 4, 6, 8, 10, 1, 3, 5, 7, 9.
10×90 = 900

15)977 - - rem. 2, 2, 2, &c. therefore, impossible.

Ex. 3. Suppose divisors, 5, 9, 11.
And remainders, 2, 8, 0.

9) 2 - - rem. 2, 7, 3, 8.
3×5 = 15

11)17 - - rem. 6, 7, 8, 9, 10, 0.
5×45 = 225

242 Ans.

THEOREMS FROM WHICH THE FOREGOING OPERATIONS MAY BE DEDUCED.

Theor. 1. When any number divided by another leaves a remainder, then the dividend increased by the divisor, or by any multiple thereof, and the sum divided by the same divisor, will leave the same remainder. For, the new dividend will consist of two parts, viz. the added number, which being equal to the divisor, or a multiple thereof, will consequently leave no remainder; and the

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original dividend, which must still leave the same remainder.

Theor. 2. If a series of numbers increasing by a common difference, be severally divided by the same divisor; then, the several remainders will also increase by a common difference, which will be equal to the remainder, when the common difference of the terms in the first series is divided by the same divisor. For, the first term of the series being divided would leave a certain remainder; and the second term being composed of the first, together with the common difference, would, if its parts were severally divided, leave two remainders, the sum of which would be the true remainder, and this would evidently differ from the first remainder by what is asserted in the theorem to be the common difference of terms in the last series.

Prob. III. To find the root of any complete *square* number, not exceeding four places of figures, by mere inspection.

Rule.] (1.) Consider the two right-hand figures of the given square, as the first period, and the remaining figure or figures, as the second period. (2.) Take the nearest root of the second period, not too great, for the first figure of the root, (place of tens,) and consider whether the square of this figure, or that of the next greater, be nearest to the aforesaid period, increased by *one*, when the adjoining right-hand figure of the next period is 5, or upwards. (3.) In the first case, when the units figure of the right-hand period is—

	1, 4, 6, 9, or 5	
The units figure of the root, will be	1, 2, 4, 3, or 5	resp.
But, in the second case, it will be	9, 8, 6, 7, or 5	

EXAMPLES.

Squares.	*Roots.*
196	14
676	26
1156	34
5401	49
9216	96
5625	75

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Prob. IV. To find the root of any complete *cube* number, not exceeding six places of figures, by inspection.

Rule.] (1.) Consider the three right-hand figures of the given cube, as the first period, and the remaining figure or figures, as the second period. (2.) Take the nearest root to the second period, not too great, for the first figure of the root. (3.) When the units figure of the given cube is - - - - - - - - 1, 4, 5, 6, or 9,
Then the units figure of the root will be
the same numbers respectively, - - 1, 4, 5, 6, or 9,
But when the units figure of the cube is
any other number - - - - - - - 2, 3, 7, or 8,
Then the units figure of the root, will be
their complements to 10 respectively. 8, 7, 3, or 2,

EXAMPLES.

Cubes.	*Roots.*
3375 - -	15
17576 - -	26
54872 - -	38
103823 - -	47
830584 - -	94

SECT. III. ARITHMETICAL AMUSEMENTS.

1. To point out the excess of any concealed number, as a person's age, above any assumed less number.

Rule (1.) Subtract, in your mind, the assumed number from the same number of places of *nines;* and desire the person to add this remainder to his concealed number; then to take away *one* from that figure of the amount which occupies the next higher place than the assumed number, and to add *one* to the units figure of said amount. Then you may confidently affirm, that the result is the excess of the concealed number above your assumed number.

Ex. 1. Suppose the concealed number to be 55, and your assumed number 33.

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99—33=66
55

121
—1
+ 1

22, excess of 55 above 33.

Ex. 2. Suppose the concealed number to be 365, and the assumed number 9.

9—9= 0
365

365
—1
+ 1

356 Excess of 365 above 9.

2. You expose to the view of a person several numbers, (let each, however, be divisible by 9, but this circumstance may be concealed) and you desire him to add secretly any two or more of those numbers together, and from the sum to strike out any one of the significant figures: you now undertake to tell him what that figure was which he struck out, on his naming the remaining figures indiscriminately.

To effect this purpose, you cast the *nines* out of the remaining figures, and what the remainder wants of nine will be the figure stricken out.

Ex. 3. Suppose the given exposed numbers to be 27, 63, 144, 315, 441, and that the first three were added together, and from the sum, 234, suppose 3 to have been stricken out, then casting the nines out of the remaining figures, 24, the remainder, 6, taken from 9 will leave 3, the figure stricken out.

3. You give any person a large number, the three places on the left hand being 900, and the units figure 3; and the rest, any figures indifferently; and you desire him to multiply it by any number consisting of two figures, and to tell you merely the two left hand figures,

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and the units figure of the product, and from these alone you may undertake to tell him his multiplier. For, the units figure of the product, with 1 or 2 prefixed, when necessary, to make it a multiple of 3, being divided by 3, will give the units figure of the multiplier; and the left-hand figure of the product, or the same increased by *one*, when the next figure is 5 or upwards, will give the second figure of the multiplier.

Note, however, that when the left-hand figure of the product is 9, or 1, not followed by a figure exceeding 7, then 1 will be the second figure of the multiplier.

Ex. 1. Suppose the given multiplicand to be 900543, and the two left-hand figures of the product announced to be 32, and the units figure 8: then $18 \div 3 = 6$, will be the units figure, and 3, the second, or left-hand figure of the multiplier.

Ex. 2. Suppose 45 to be the two left-hand figures of the product, and 3 the units figure: then 51 will have been the multiplier.

Ex. 3. Suppose 99 to be the two left-hand figures of the product, and 1 the units figure; then 17 must have been the multiplier.

Ex. 4. Suppose the two left-hand figures of the product to be 17, and the units figure 7: then (see the above note) 19 must have been the multiplier.

4. You desire any person to write down, secretly, any two numbers, one consisting of an even, and the other of an odd number of places; and to take either of the two numbers at pleasure; add to it its *reverse*, (the same number set down backwards) and divide the sum by 11. You then desire him to subtract *one* from the remainder. Should he tell you that he cannot (from which you infer that there was no remainder) then, you may pronounce with certainty, that he chose the number of even places; and vice versa.

☞ The above rule will never fail except when the number of *odd* places is chosen, and that it should happen to be divisible by 11; for then the sum of this number and its reverse will also be divisible by 11.

Ex. Suppose the two numbers chosen to be 365 and 51.

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365	51
563	15
11)928(84 - - rem. 4.	11)66(6 - - 0 rem.

5. You write down a number, and conceal it under cover, or place it in the hands of a person in company. Then you desire any person to write down three lines of figures, three in each line, under each other; and you propose to write down instantly, three other lines under the former; so that the sum of the six lines when added together, shall be the same as the number you had previously written down.

To effect this, you write down, in the same order, the complements of the upper numbers respectively to 9; for then, the sum of each column of figures will be 3 times 9, and, consequently, the sum will always be 2997

Ex. 345 } Given.
729 }
653 }
654 } Subscribed.
270 }
436 }

2997

The above may be varied, so as to prevent the rule from being discovered, by taking the complement of some of the numbers in the upper columns to any other numbers, as to 7, 8, 10, according to your previous determination; for then the resulting sum will be varied accordingly. For instance, if I determine to take the complement to 10, of some figure in the units column, and the complement to 8 of some one in the tens column; then, the sum would become 2988, &c.

6. You propose to tell the age of any person in company. To accomplish this, you desire him to divide his age by 4 and by 3, and to tell you the respective remainders. Let the two remainders be announced 2 and 1. I then find by Problem II. foregoing, the least number, and

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thence a series of numbers, which will have the property required.

These numbers will be found to be 10, 22, 34, 48, &c.

3)2 - - - rem. 2, 0, 1.

2×4= 8

10,

4×3= 12

22, 34, 48, &c.

Then, if I can certainly guess within less than 12 years of the age, which in almost all cases may be done, I can with certainty tell the age. For instance, if I am sure that the age must be between 20 and 30, then I pronounce with confidence that the age in question is 22.

7. You propose to a companion as the subject of a prize or trial of skill, to add, alternately, any numbers at pleasure, not exceeding a given limit, and that the one who shall first arrive at any given sum, previously agreed on, shall gain the prize or be considered as the victor.

(1.) Let the limit of a single addition be not less than 2, nor more than 5, and the final sum or amount 50.— In this case, if you take the first number, you may, with the proper exercise of skill, certainly gain the prize.— To effect this you begin with the number 1, the remainder, when 50, is divided by (2+5) 7; then let your companion add what number he will, within the stipulated limit, you may always, taking the supplement of his number, to 7, by your second addition, get 8, by your third, 15, and so on, to 50, the ultimate sum. And even if your companion should begin, one false step would still put the prize in your power.

Any other similar conditions being proposed, a little consideration will enable you to discover the law of success.

A great variety of other arithmetical amusements might be added, but the limits prescribed will not admit of their insertion.

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ALGEBRAIC DEMONSTRATIONS OF THE PRINCIPAL RULES OR THEOREMS IN ARITHMETIC.

I. IN PROPORTION.

1. If four quantities, a, b, c, d, be directly proportional, viz. $a : b :: c : d$; then the product of the two extremes will equal the product of the two means - - - $ad = bc$.

For, $\frac{a}{b} = \frac{c}{d}$ per def. of ratios.

$\times bd$, then $\frac{abd}{b} = \frac{cbd}{d}$; or $ad = bc$.

2. If two products be equal, viz. $ad = bc$, then their factors will be reciprocally proportional - - - $a : b :: c : d$.

For, since $ad = bc$, per supposition.

$\div bd$, then $\frac{ad}{bd} = \frac{bc}{bd}$; or $\frac{a}{b} = \frac{c}{d}$ $\therefore a : b :: c : d$.

3. If four quantities be proportional, viz. $a : b :: c : d$.

Then (1.) $a : c :: b : d$.
(2.) $b : a :: d : c$.
(3.) $c : a :: d : b$.
(4.) $a+b : b :: c+d : d$.
(5.) $a-b : b :: c-d : d$.
(6.) $a : a+b :: c : c+d$.
(7.) $a : a-b :: c : c-d$, &c. &c.

For, in all the above proportions, the product of the two extremes will = the product of the two means, and in their lowest terms will be $ad = bc$.

4. If $a : b :: c : d$
$c : d :: e : f$
$e : f :: g : h$; then $a : b :: g : h$.

For, $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h}$ $\therefore \frac{a}{b} = \frac{g}{h}$ and $a : b :: g : h$.

All the rules, or theorems, relating to proportion, in arithmetic, may readily be derived from the above.—Thus, from three given terms, a, b, c, to find a fourth

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proportional, d, we have the equation $ad = bc$, and this divided by a, will give $d = \frac{bc}{a}$. In words—multiply the second and third terms together, and divide the product by the first, and the quotient will be the fourth, or term required.

II. IN ALLIGATION ALTERNATE.

Let a, b, = the respective rates of two ingredients mixed together, and m = the mean rate of the mixture; and suppose $a > b$. Then, per rule.

$$m \begin{cases} a \\ b \end{cases} \begin{matrix} m - b = \text{the quantity of ingredient } a, \\ a - m = \text{the quantity of ingredient } b; \end{matrix}$$

and $(m - b) \times a + (a - m) \times b = am - bm$; which also $= [(m - b) + (a - m)] \times m$. That is, the several quantities in the mixture multiplied by their respective rates = the whole quantity multiplied by the mean rate; which being evidently the case, demonstrates the truth of the rule. In the same manner the truth of the rule may be demonstrated when applied to three or more ingredients.

III. IN THE INVESTIGATION OF THE RULE FOR THE EXTRACTION OF THE SQUARE ROOT.

Let n = any number whose square root is to be found.
r = the nearest root, but less than the true.
x = what r wants of the true root, and $r + x$ = the true root.

Then, $(r+x)^2 = n$
i. e. $r^2 + 2rx + x^2 = n$
$2rx + x^2 = n - r^2$
$x = \frac{n - r^2}{2r + x}$,

which, expressed in words, would give the general rule or theorem, for extracting the square root; $n - r^2$ being called the *resolvend*, and $2r$, the *defective divisor*, by means of which, the next figure of the root (x) is to be found; and this added to the defective divisor, will give the *complete* divisor, $2r + x$.

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IV. IN THE INVESTIGATION OF THE RULE FOR THE EXTRACTION OF THE CUBE ROOT.

Let n = any number whose cube root is to be found.
r = the nearest root, but less than the true.
x = what r wants of the true root.

Then, $(r+x)^3 = n$
i. e. $r^3+3r^2x+3rx^2+x^3 = n$
$3r^2x+3rx^2+x^3 = n-r^3$
$$x = \frac{n-r^3}{3r^2+x^3+3rx};$$

which, expressed in words, would give the general rule for extracting the cube-root; $n - r^3$ being called the *resolvend*, and $3r^2$, the *defective divisor*, by means of which the next figure of the root (x) is to be found; and then $x^2 + 3rx$ added to to the defective divisor will give the complete divisor $3r^2 + x^2 + 3rx$.

V. IN THE INVESTIGATION OF THE THEOREMS IN ARITHMETICAL PROGRESSION.

Let l = the least term,
g = the greatest term,
d = the common difference,
$\mathcal{N}$ = the number of terms,
n = the number of terms less *one* = $\mathcal{N} - 1$,
s = the sum of the series.

Then the terms of the series will be as follows,—

Ascending, $l, +l+d, +l+2d, +l+3d - - - + l + nd = s$
Descending, $g, +g-d, +g-2d, +g-3d - - +g - nd = s$
By add'n, $g+l, +g+l, +g+l - - - - - +g+l = 2s$.
That is, $g+l$ taken $\mathcal{N}$ times, or $(g+l) \times \mathcal{N} = 2s$.
Hence $s = (g+l) \times \frac{1}{2}\mathcal{N}$. Theor. 1.
And $g = l + nd$. Theor. 2.

From the above theorems, which may be considered as fundamental, it follows, that of the *five* particulars, l, g, d, n, s, any *three* being given, the other two may be

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found; and thus, by an easy algebraic process, all the appropriate theorems may be investigated*.

VI. IN THE INVESTIGATION OF THEOREMS IN GEOMETRICAL PROGRESSION.

Let l = the least term,
g = the greatest term,
R = the common ratio,
r = the common ratio less *one* ($=R-1$,)
$\mathcal{N}$ = the number of terms,
n = the number of terms less one, ($=\mathcal{N}-1$,)
s = the sum of the series,
L = logarithm of.

Then the terms of the series will be as follows—

$$l, + lR, + lR^2 + lR^3 - - - - - - + lR^n = s.$$

Hence it appears that the terms of this series are *continual* proportionals; namely, $l : lR :: lR : lR^2 : lR^3$, &c. all the terms being antecedents except the last, and all consequents except the first. Therefore, the sum of all the antecedents will $= s - g$, and the sum of all the consequents $= s - l$: but as one of the antecedents is to its consequent, so is the sum of all the antecedents, to the sum of all the consequents†.

That is, $l : lR :: s - g : s - l$.

Hence $s = \frac{Rg - l}{r}$. Theor. 1. And from the above series it appears,

That $g = lR^n$. Theor. 2.

* In the theorems in arithmetical progression, p. 18, for n, in theor. 7 and 8, read $\mathcal{N}$; and in theor. 9, dele $\div$.

† Suppose r, r^2, r^3, r^4, to be a series of continual proportionals; then, I say, $r : r^2 :: r + r^2 + r^3 : r^2 + r^3 + r^4$; for the product of the two extremes is identical with the product of the two means.

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Now, from these two equations, any three of the five particulars, l, g, r, n, s, being given, theorems may be deduced for finding the other two*.

VII. IN ANNUITIES AT SIMPLE INTEREST.

Let u = the annuity,
T = the time of continuance in years,
$t = T - 1$,
r = the interest of one dollar, &c. in one year,
a = the amount in t years,
p = the present worth of the annuity.

Then, Tu = the amount of annuity, without interest, at the end of T years.

o = the interest due on last annuity,
ur = ditto - - - - - - last but one,
$2ur$ = ditto - - - - - - last but two,
$3ur$ = ditto - - - - - - last but three,
and tur = ditto - - - - - - first annuity.

Hence, it appears, that the several interests form a rank of terms, in arithmetical progression, whose sum, per Art. V., Theor. 1, $= (tr+2) \times \frac{1}{2}T$, and therefore $a = (tr+2) \times \frac{1}{2}T + Tu$.

But, $p\,Tr + p$ would = the amount of principal p for the same time and rate: and hence, $p\,Tr + p = (tr+2) \times \frac{1}{2}T + Tu$; or, $p = [(tr+2) \times \frac{1}{2}T + Tu] \div (Tr+1.)$

It follows then, that, from the two above equations, any three of the five particulars, u, t, r, a, p, being given, the other two may be found.

VIII. IN COMPOUND INTEREST.

Let p = the principal, or sum at interest,
t = the time of its continuance, in years,
R = the ratio, or amount of one dollar, &c. in one year,
a = the amount of principal in t years, at the given ratio R.
L = logarithm of.

* In p. 21, there is a typographical error; l, before logarithm of, instead of L.

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Hence, since R = the am. of one doll. &c. in 1 year,
R^2 = ditto - - - - - - - - - in 2 years,
R^3 = ditto - - - - - - - - - in 3 years,
and R^t = ditto - - - - - - - - - in t years.
Therefore, $a = fR^t$; or, $La = Lf + LR \times t$. Theor. 1.

From the above equation, it follows, that any three of the four particulars, f, t, R, a, being given, the fourth may be found.

IX. IN ANNUITIES IN ARREAR AT COMPOUND INTEREST.

Let u = the annuity,
R = the ratio &c. as in compound interest,
$r = R - 1$,
t = the time for which the annuity is in arrear.
$m = R^t - 1$,
a = the amount of annuity at the end of t years,
P = the present worth of annuity.
Then u = the amount of last annuity,
uR = ditto of last but one,
uR^2 = ditto of last but two,
uR^3 = ditto of last but three,
And uR^t = ditto of first annuity.

These terms evidently constitute a series in geometrical progression, whose sum, by a foregoing theorem

$$= \frac{uR^t - u}{r}$$

Hence $a = \frac{uR^t - u}{r}$, or $La = Lm + Lu - Lr$.

But $a = fR^t$; hence $fR^t = \frac{uR^t - u}{r}$: or $f = \frac{uR^t - u}{rR^t}$;

or, $Lf = Lm + Lu - Lr - LR \times t$.

Therefore, from the two above equations it follows, that any three of the five particulars u, R, t, a, f being given, the other two may be found.

X. IN REVERSIONARY ANNUITIES, AT COMPOUND INTEREST.

Let u = the annuity,
R = ratio &c. as before,

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r = $R-1$,
T = the time of reversion,
t = the time of continuance,
p = the present worth of reversion,
m = R^t-1,
w = $T+t$,
L = logarithm of.

From the last article it appears that $\frac{uR^t-u}{rR^t}$ would = the worth of the annuity, purchased at the time of its commencement. And from article viii. we would have $\frac{uR^t-u}{rR^t} \div R^T \left(=\frac{uR^t-u}{rRw}\right)$ = the principal which would produce that amount, that is, the present worth of the reversion. Hence $p = \frac{uR^t-u}{rRw}$; or, $Lp = Lm + Lu - LR \times w$. And therefore, from this equation, any three of the four particulars u, R, $\left\{\frac{T}{t}\right\}$, p being given, the other two may be found.

☞ In the above articles, respecting progressions, interest, and annuities, the fundamental equations or theorems *only* have been investigated; it being left as no unprofitable exercise for the young student, when he shall have acquired a competent knowledge of algebra, to investigate the rest.

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ANSWERS TO THE EXAMPLES

IN THE FOREGOING

PART OF ARITHMETIC.

IN INVOLUTION.

1. - - - - - - - - - 1099511627776
2. - - - - - - - - - 205891132094649
3. - - - - - - - - - 281474976710656
4. - - - - - - - - - 68719476736

IN EVOLUTION.

1. - - - - - - - - - - - - 8.15475
2. - - - - - - - - - - - - 6.479179
3. - - - - - - - - - - - - 74.76
4. - - - - - - - - - - - - 69.13
5. - - - - - - - - - - - - $\frac{7}{10}$
6. - - - - - - - - - - - - .866
7. - - - - - - - - - - - - 12.5893
8. - - - - - - - - - - - - 1.150160

IN ARITHMETICAL PROGRESSION.

1. - - - - - - - - - - - - 82 and 1161
2. - - - - - - - 14400 ft. and 18.16 sec.
3. - - - - - - - - - - - - $10\frac{40}{99}$ ft.
4. - - - - - - - - - - - - $13\times6=78$
5. - - - - - - - $(13+156)\times36=6084$

IN GEOMETRICAL PROGRESSION.

1. - - - - - - - - - - 49152 and 98301
2. - - - - - - - - - - - - 4 and 7

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3. - - - - - - - - - - - - - 4800
4. - - - - - - - - - - - - - $1+10^{97}$

IN THE USE OF LOGARITHMS.

1. - - - - - - - - - - - - - 205505.5
2. - - - - - - - 17.13615 and .058356
3. - - - - - - - - - - - - - 6.0881
4. - - - - - - - - - - - - - 1.06
5. - - - - - - - - - - - - - .186198
6. - - - - - - - - - - - - - .997518
7. - - - - - - - - - - - - - 39.6987
8. - - - - - - - - - - - - - .488365

IN COMPOUND INTEREST.

1. - - - - - - - - - - - - - $631.24
2. - - - - - - - - - - - - - 1323.84
3. - - - - - - - - - - - - - 1149.07
4. - - - - - - - - - - - - - 540.45
5. - - - - - - - - - - - - - 1575.13
6. - - - - - - - - - - - - - 5 per cent.
7. - - - - - - - - - - - - - 4 years.
8. - - - - - - - - - - - - - 11.9 years.

IN ANNUITIES IN ARREAR, AT SIMPLE INTEREST.

1. - - - - - - - - - - - - - - $840
2. - - - - - - - - - - - - - - 121.06
3. - - - - - - - - - - - - - 6 per cent.
4. - - - - - - - - - - - - - 5.05 years.

IN PURCHASING ANNUITIES, AT SIMPLE INTEREST.

1. - - - - - - - - - - - - - - $176.72
2. - - - - - - - - - - - - - - 45.85
3. - - - - - - - - - - - - - 6 per cent.
4. - - - - - - - - - - - - - 5 years.

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IN REVERSIONARY ANNUITIES, AT SIMPLE INTEREST.

1. - - - - - - - - - - - - - $507.57
2. - - - - - - - - - - - - - 340.

IN ANNUITIES IN ARREAR, AT COMPOUND INTEREST.

1. - - - - - - - - - - - - - $1293.04
2. - - - - - - - - - - - - - 500
3. - - - - - - - - - - - - - 7 years.
4. - - - - - - - - - - - - - 141.46
5. - - - - - - - - - - - - - 200.

IN REVERSIONARY ANNUITIES, AT COMPOUND INTEREST.

1. - - - - - - - - - - - - - $1523.35
2. - - - - - - - - - - - - - 420.50
3. - - - - - - - - - - - - - 3 yrs. 10 mo.

IN REVERSIONARY PERPETUITIES, AT COMPOUND INTEREST.

1. - - - - - - - - - - - - - $8900.
2. - - - - - - - - - - - - - 289.80
3. - - - - - - - - - - - - - 3 years.

IN LIFE ANNUITIES, IN WHICH NOT MORE THAN TWO LIVES ARE CONCERNED.

1. - - - - - - - - - - - - - $5340.
2. - - - - - - - - - - - - - 3546.67
3. - - - - - - - - - - - - - 2873.15
4. - - - - - - - - - - - - - 1273.53
5. - - - - - - - - - - - - - 10335.36
6. - - - - - - - - - - - - - 8388.40
7. - - - - - - - - - - - - - 402.98
8. - - - - - - - - - - - - - 231.86

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IN LIFE ANNUITIES, IN WHICH MORE THAN TWO LIVES ARE CONCERNED.

1. - - - - - - - - - - - - $1620.60
2. - - - - - - - - - - - - 760.63
3. - - - - - - - - - - - - 154.

IN PERMUTATION.

1. - - - - - - - - - - - - - 40320
2. - - - - - - - - - - - - - 40320

IN COMBINATIONS.

1. - - - - - - - - - - - - - 84.
2. - - - - - - - - - - - - - 181440
3. - - - - - - - - - - - - - 4782969
4. - - - - - - - - - - - - - 69760
5. - - - - - - - - - - - - - 756
6. - - - - - - - - - - - - - 336
7. - - - - - - - - - - - - - 70
8. - - - - - - - - - - - 218322000.

IN ARITHMETIC OF CHANCES.

1. - - - - - - - - - - - $\frac{1}{6} - \frac{1}{36} = \frac{5}{36}$
2. - - - - - - - - - - - $\frac{1}{66}$
3. - - - - - - - - - - - $\frac{21364}{1860867}$

FINIS.

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XII Splendid suggestion about notation — ~~not necessary~~ th
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It still adheres to rules, yet
gives evidence of being
written by a thoughtful man.
This certainly is a "treatise"

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